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CONCENTRATED FORCE PROBLEMS IN TRANSVERSE ISOTROPY.(U)

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G.M. TONEATTO

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Basic singular solutions for the equations of elasticity in a three-dimensional transversely isotropic domain are generated and studied.

A rational approach is then presented for the generation of the solution of various concentrated-load problems in the half-space. In particular, a previously unknown solution for a concentrated load in the half-space parallel to the free surface is derived and discussed.

A completeness proof for a stress-function approach for problems in three dimensional transverse isotropy is also presented.

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6 CONCENTRATED FORCE PROBLEMS IN TRANSVERSE ISOTROPY

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1. INTRODUCTION

1.1 Objective and Scope

The solution of problems of elastic stress distributions in isotropic materials has been considered in great detail in the literature but, considering the recognized anisotropy of virtually all materials, relatively little work has been done on similar problems involving anisotropy. Once the assumption of isotropy is discarded, analysis of any three-dimensional problem becomes significantly more difficult. This is, of course, due to the involvement of more than two elastic constants.

It is of note that a proportionately large number of general solutions can be found for materials which are "transversely isotropic" and have a hexagonal close-packed crystalline structure. These materials have five elastic constants. Although many solutions exist for this special case of anisotropy, techniques vary widely and a clear attack has been difficult to devise.

The objective of this study is to develop an efficient and useful technique for solving problems in a "transversely isotropic" medium and, more precisely, to present a solution for the problem of a tangential force applied beneath the surface of a "transversely isotropic" half-space.

1.2 History and Concept of Singularities

In 1872, according to A.E.H. Love (1926)*, E. Betti first applied the

---

* An author's name followed by a date of publication refers to entries in the List of References.

method of singularities to the theory of elasticity. Betti was able to deduce an average strain formula for forces acting on an isotropic body and found that this method was more effective than the method of series as a tool for solving force transmission problems.

Lord Kelvin's solution of the problem of a concentrated force in an unbounded isotropic medium when combined with Betti's method of singularities allowed a family of singular solutions to be generated. In an isotropic medium, these singular solutions, generated by synthesis of the Kelvin Solution, have been designated by A.E.H. Love (1926) as "nuclei of strain". Such nuclei are obtained through superposition and limiting processes of differentiation and integration. Therefore, the derivative of the Kelvin Solution in the direction of its force, generates what Love called a "double force without moment" and is often referred to as a force doublet. Moreover, the derivative of the Kelvin Solution perpendicular to the direction of its force, generates a "double force with moment". Superposition of three mutually perpendicular "double forces without moment" creates a "center of dilatation" or a "center of compression", depending on the orientation of the forces, while the superposition of two perpendicular "double forces with moment" which share the same axis, generate a "center of rotation". Clearly, all these solutions are singular at a point.

Now, it is well to note that the stresses of the Kelvin solution become infinite as $\frac{1}{R^2}$ as R, the distance from the observation point to the point of application, approaches zero. This is denoted $O(\frac{1}{R^2})$. Further differentiation will yield solutions which are of yet higher order at the point of application.

In addition to solutions which are singular at a point, line singularities can also be generated. For example, integration of a solution for a center of dilatation along, say, the z-axis from the origin to infinity, yeilds a solution for a "line of dilatation". When this limiting process is continued, "lower order singularities" are obtained.

1.3 History and Concept of Half-Space Problems

In the investigation of problems involving a plane boundary on an infinite isotropic medium ("half-space"), J. Boussinesq first presented solutions for normal tractions and tangential displacements or normal displacements and tangential tractions. At about the same time, V. Cerruti obtained the same results by another method. For this reason the problem of the half-space is often referred to as the "problem of Boussinesq and Cerruti".

Soon afterwards, J.H. Michell (1900) obtained a solution for this problem by yet another method. Love (1926) later presented his solutions of the Boussinesq (concentrated force acting perpendicular to a plane surface of an isotropic half-space) and the Cerruti (concentrated force acting tangential to the plane surface of an isotropic half-space) problems in which he used a method of solution involving singularities and refined by C. Somigliana and G. Lauricella before the turn of the century.

Careful analysis of Michell's (1901) work indicates that he also solved the problem of a vertical force acting at a point within a semi-infinite isotropic solid. More recently, R.D. Mindlin (1936) rediscovered this solution which now bears his name.

1.4 History and Concept of "Transverse Isotropy"

Materials or bodies which possess an axis of symmetry such that perpendicular to any point along an axis the material behaves in the same manner regardless of direction, are known as being "transversely isotropic". This anisotropic material is sometimes called "hexagonal aeolotropic" or "aeolotropic elastic" in the literature, but to avoid confusion, only Love's (1926) term, "transversely isotropic" will be used here.

In order to explain the meaning of a "transversely isotropic" medium, let us adopt the notation:

$$\tau_1 = \tau_{xx} \, , \tau_2 = \tau_{yy} \, , \tau_3 = \tau_{zz} \, , \tau_4 = \tau_{yz} \, ,$$

$$\tau_5 = \tau_{zx} \, , \tau_6 = \tau_{xy} \, , e_1 = e_{xx} \, , e_2 = e_{yy},$$

$$e_3 = e_{zz} \, , e_4 = e_{yz} \, , e_5 = e_{zy} \, , e_6 = e_{xy}. \qquad (1.4.1)$$

where τ and e represent the Cartesian components of stress and "infinitesimal strain" respectively. The strains are now defined by

$$e_{xx} = \frac{\partial U_1}{\partial x} \, ,\ldots,;$$

$$e_{yy} = \frac{\partial U_2}{\partial y} \, ,\ldots,,$$

$$e_{yz} = \frac{\partial U_2}{\partial z} + \frac{\partial U_3}{\partial y} \, ,\ldots,$$

where $[U_1, U_2, U_3]$ are the Cartesian scalar components of the displacement vector. The general linear stress-strain law now becomes:

$$\tau_i = c_{ij} e_j \tag{1.4.2}$$

where i and j range over the integers one to six. A.E.H. Love (1926) noted that for a homogeneous medium the c_{ij} are constants and he showed that:

$$c_{ij} = c_{ji} \, , \tag{1.4.3}$$

is a necessary and sufficient condition for the existence of a strain-energy function $W(e_1, \ldots, e_6)$ such that

$$\tau_i = \frac{\partial W}{\partial e_i} \tag{1.4.4}$$

and that

$$W = \frac{1}{2} c_{ij} e_i e_j \tag{1.4.5}$$

Love then imposed the condition of transverse isotropy and noted that

$$C_{ij} = 0 \quad (i = 1,2,3;\ j = 4,5,6),\ (i = 4,5,6;\ j \neq i)$$

$$C_{11} = C_{22} \, , \; C_{13} = C_{23} \, , \; C_{44} = C_{55} \, , \; 2C_{66} = C_{11} - C_{12} \tag{1.4.6}$$

Adopting the notation used by Eubanks and Sternberg (1954),

$$C_{11} = a \; , \; C_{33} = \bar{a} \; , \; C_{44} = \mu \; , \; C_{66} = \bar{\mu} \; ,$$

$$C_{12} = a - 2\bar{\mu} \; , \; C_{13} = b,$$

it can be written (letting U_1, U_2 and U_3 be the Cartesian components of displacement) that:

$$\sigma_{xx} = a \frac{\partial U_1}{\partial x} + (a - 2\bar{\mu}) \frac{\partial U_2}{\partial y} + b \frac{\partial U_3}{\partial z}$$

$$\sigma_{yy} = (a-2\bar{\mu}) \frac{\partial U_1}{\partial x} + a \frac{\partial U_2}{\partial y} + b \frac{\partial U_3}{\partial z}$$

$$\sigma_{zz} = b \frac{\partial U_1}{\partial x} + b \frac{\partial U_2}{\partial y} + \bar{a} \frac{\partial U_3}{\partial z}$$

$$\sigma_{xz} = \mu \frac{\partial U_1}{\partial z} + \mu \frac{\partial U_3}{\partial x}$$

$$\sigma_{yz} = \mu \frac{\partial U_2}{\partial z} + \mu \frac{\partial U_3}{\partial y}$$

$$\sigma_{xy} = \bar{\mu} \frac{\partial U_2}{\partial x} + \bar{\mu} \frac{\partial U_1}{\partial y} \tag{1.4.8}$$

These relations involve five independent constants. The physical meaning of μ and $\bar{\mu}$ is clear.

Eubanks and Sternberg (1954) also show that for the positive definiteness of W, the following necessary and sufficient conditions exist:

$$a > 0 \text{ , } \bar{a} > 0 \text{ , } \mu > 0 \text{ , } \bar{\mu} > 0 \text{ ,}$$

$$a\,\bar{a} - b^2 - \bar{a}\,\bar{\mu} > 0. \tag{1.4.9}$$

or similarly:

$$E > 0 \quad \bar{E} > 0 \quad \mu > 0 \quad \bar{\mu} > 0$$

$$-1 < \nu < 1,\ 1 - \nu > \frac{2\bar{E}\,\bar{\nu}^2}{E} \tag{1.4.10}$$

where

$$E = \frac{4\bar{\mu}\,(a\bar{a} - b^2 - a\,\bar{\mu}}{a\bar{a} - b^2}$$

$$\bar{E} = \frac{a\bar{a} - b^2 - \bar{a}\,\bar{\mu}}{a - \bar{\mu}}$$

$$\nu = \frac{a\bar{a} - b^2 - 2\bar{a}\,\bar{\mu}}{a\bar{a} - b^2}$$

$$\bar{\nu} = \frac{2b\bar{\mu}}{a\bar{a} - b^2} \qquad (1.4.11)$$

The respective strain equations reduce to the following:

$$e_x = + \frac{\sigma_x}{E} - \frac{\sigma_y \nu}{E} - \frac{\sigma_z \bar{\nu}}{E}$$

$$e_y = - \frac{\sigma_x \nu}{E} + \frac{\sigma_y}{E} - \frac{\sigma_z \bar{\nu}}{E}$$

$$e_z = - \frac{\sigma_x \bar{\nu}}{E} - \frac{\sigma_y \bar{\nu}}{E} + \frac{\sigma_z}{\bar{E}} \qquad (1.4.12)$$

It is interesting to examine the states of strain which result in some simple uni-axial loading (see Table 1.1).

Table 1.1 STRAINS: UNI-AXIAL LOADING

	$\sigma_x = 1$ $\sigma_y = \sigma_z = 0$	$\sigma_y = 1$ $\sigma_x = \sigma_z = 0$	$\sigma_z = 1$ $\sigma_x = \sigma_y = 0$
e_v	$\frac{1}{E}$	$-\frac{\nu}{E}$	$-\frac{\bar{\nu}}{E}$
e_y	$-\frac{\nu}{E}$	$\frac{1}{E}$	$-\frac{\bar{\nu}}{E}$
e_z	$-\frac{\bar{\nu}}{E}$	$-\frac{\bar{\nu}}{E}$	$\frac{1}{E}$

It is also of interest to examine values of the constants mentioned earlier for some materials which exhibit transversely isotropic behavior. Magnesium, zinc and cadmium are three such materials (Elliott (1948) has already worked with the properties of magnesium and zinc). Using data from F. Seitz and T.A. Read (1941) and C.S. Barrett (1943) and using the notation of Eubanks and Sternberg (1954), the values in Table 1.2 were calculated for these materials.

Table 1.2 ELASTIC CONSTANTS

	MAGNESIUM	ZINC	CADMIUM	UNITS
a	5.6493 $\times 10^{11}$	16.3593 $\times 10^{11}$	12.0587 $\times 10^{11}$	dynes/cm^2
$\bar{a}$	5.8734 $\times 10^{11}$	6.2926 $\times 10^{11}$	5.1326 $\times 10^{11}$	"
b	1.8103 $\times 10^{11}$	5.1666 $\times 10^{11}$	4.4197 $\times 10^{11}$	"
μ	1.6807 $\times 10^{11}$	3.7879 $\times 10^{11}$	1.8519 $\times 10^{11}$	"
$\bar{\mu}$	1.6667 $\times 10^{11}$	6.8493 $\times 10^{11}$	3.6232 $\times 10^{11}$	"
E	4.4843 $\times 10^{11}$	11.9048 $\times 10^{11}$	8.1301 $\times 10^{11}$	"
$\bar{E}$	5.0505 $\times 10^{11}$	3.4843 $\times 10^{11}$	2.8169 $\times 10^{11}$	"
ν	.3453	-.1310	.1220	—
$\bar{\nu}$	.2018	.9286	.7561	—

Use of Tables 1.1 and 1.2 allows one to calculate the values listed in Table 1.3 below:

Table 1.3 STRAIN RATIOS

	DIRECTION OF UNI-AXIAL LOADING		
	X	Y	Z
MAGNESIUM	$\frac{e_y}{e_x} = -.3453$	$\frac{e_x}{e_y} = -.3453$	$\frac{e_x}{e_z} = -.2273$
	$\frac{e_z}{e_x} = -.2018$	$\frac{e_z}{e_y} = -.2018$	$\frac{e_y}{e_z} = -.2273$
ZINC	$\frac{e_y}{e_x} = +.1310$	$\frac{e_x}{e_y} = +.1310$	$\frac{e_x}{e_z} = -.2718$
	$\frac{e_z}{e_x} = -.9286$	$\frac{e_z}{e_y} = -.9286$	$\frac{e_y}{e_z} = -.2718$
CADMIUM	$\frac{e_y}{e_x} = -.1220$	$\frac{e_x}{e_y} = -.1220$	$\frac{e_x}{e_z} = -.2620$
	$\frac{e_z}{e_x} = -.7561$	$\frac{e_z}{e_y} = -.7561$	$\frac{e_y}{e_z} = -.2620$

Observation of the values listed in Table 3 leads one to expect Magnesium to be the most nearly isotropic of the three materials. Furthermore, the fact that the strain ratios for both zinc and cadmium fall outside the commonly encountered range for Poisson's ratio in isotropic materials, (in addition to the large numerical differences between E and $\overline{E}$ and, ν and $\overline{\nu}$ for these materials) leads one to expect significantly different behavior for zinc and cadmium than for some isotropic material or nearly-isotropic material like magensium. More discussion of these materials will be given in Chapter 5.

The equations of equilibrium in terms of stresses are not modified for an anisotropic material. On the other hand, in terms of displacements, they have the following form:

$$a\frac{\partial^2 U_1}{\partial x^2} + \overline{\mu}\frac{\partial^2 U_1}{\partial y^2} + \mu\frac{\partial^2 U_1}{\partial z^2} + (a-\overline{\mu})\frac{\partial^2 U_2}{\partial x \partial y} + (b+\mu)\frac{\partial^2 U_3}{\partial x \partial z} = 0$$

$$a\frac{\partial^2 U_2}{\partial y^2} + \overline{\mu}\frac{\partial^2 U_2}{\partial x^2} + \mu\frac{\partial^2 U_2}{\partial z^2} + (a-\overline{\mu})\frac{\partial^2 U_1}{\partial x \partial y} + (b+\mu)\frac{\partial^2 U_3}{\partial y \partial z} = 0$$

$$\overline{a}\frac{\partial^2 U_3}{\partial z^2} + \mu\frac{\partial^2 U_3}{\partial x^2} + \mu\frac{\partial^2 U_3}{\partial y^2} + (b+\mu)\frac{\partial^2 U_1}{\partial x \partial z} + (b+\mu)\frac{\partial^2 U_2}{\partial y \partial z} = 0 \qquad (1.4.13)$$

These equations, of course, assume the standard form in the case of isotropy where:

$$a = \overline{a} = 2\mu + \lambda = \frac{2(1-\nu)\mu}{1-2\nu}$$

$$b = \lambda = \frac{2\nu\mu}{1-2\nu}$$

$$\mu = \bar{\mu} = \frac{E}{2(1+\nu)} \qquad (1.4.14)$$

A degeneracy is created for the special case of $(b+\mu) = 0$ which should be studied separately.

Transversely isotropic material has received reasonable attention since I. Fredholm (1900) treated certain special cases of anisotropy. He gave an implicit expression for Green's function for general anisotropic media and also solved the problem of an infinite transversely isotropic material acted upon by a concentrated force at a point in the medium perpendicular or parallel to the axis of elastic symmetry. Fredholm also implies the curl-potential solution of the equilibrium equations for anisotropy.

Michell (1900), in his somewhat obscurely written paper, presented the primary foundation for the formulation for the general solution of the problem of the half-space in transverse isotropy for arbitrarily prescribed body forces, surface tractions and surface displacements. All of the singular problems for the half-space including those of Boussinesq, Cerruti, Mindlin and the solution presented here, must be considered to be subsumed in the formulations presented in Michell's remarkable paper. A summary of Michells work is contained in Niedenfuhr (1964).

S.G. Lekhnitsky (1940) considered problems of torsionless axisymmetry. He derived particular solutions by using a generalized form of Love's Function and generated equations of equilibrium in terms of displacements with a single stress function which satisfied a fourth order partial differential equation. R.A. Eubanks and E. Sternberg (1956) later provided the completeness proof of Lekhnitsky's work.

After Lekhnitsky's presentation, H.A. Elliott (1948) solved the torsionless rotationally-symmetric three-dimensional field equations in terms of two stress functions, each of which satisfied a second-order partial differential equation. Both Lekhnitsky's and Elliott's contributions are described by J.N. Goodier and P.G. Hodge (1958). Although Elliott did not apply his method to a non-axisymmetric field, the general solution he obtained is of relatively great importance. Elliott's approach was also applied by L.E. Payne (1954), D.S. Berry and T.W. Sales (1961), and A.H. England (1962) to problems with or without axisymmetry. B. Sharma (1958) and, Z Massakaowska and W. Nowacki (1958) extended Elliott's approach to problems in thermoelasticity.

Shortly after Elliott presented his results, R.T. Shield (1951) solved the problem of an isolated line force uniformly distributed through a transversely isotropic plate and acting parallel to the faces of the plate. In the same paper, Shield solved the problem of a sub-surface vertical force in a transversely isotropic medium, as well as the punch and flat elliptical crack problems for such media. A treatise by A.E. Green and W. Zerna (1954) outlines both Elliott's and Shield's contributions.

A closed form solution of Green's function for transversely isotropic materials was obtained by E. Kröner (1953). This problem had also been studied by I.M. Lifshitz and L.N. Rozentsvieg (1947).

H.-C. Hu (1954,1956) solved the normal and tangential surface load problems as well as the rigid stamp problem and the problem of the bending of a thin elastic plate lying on a transversely isotropic half-space. He was apparently the first to apply the implied Fredholm curl-potential solution for the solving of non-axisymmetric problems in transverse isotropy. A.S. Lodge (1955) interestingly, also found the third potential-function solution but did not follow it up further.

The use of Fredholm's work and Kroner's closed form solution, allowed T.C. Woo and R.T. Shield (1962), (who were actually dealing with the general theory of small elastic deformations superimposed on large elastic deformations) to also solve the significant problems of a concentrated surface force acting perpendicular or parallel to the plane boundary of a semi-infinite transversely isotropic medium. These same problems were again solved by Y.-C. Pan and T.-W. Chou (1976) except that the results were expressed in a slightly different form.

The last ten years have seen the publication of many works on transversely isotropic elasticity. A discussion of all of these is not within the scope of this study.

1.5 Organization of the Study

In Chapter 2 a displacement-potential approach is presented for the solution of problems in a transversely isotropic region and its completeness for the general three-dimensional problem is proved. Some consequences of this presentation are then discussed.

Chapter 3 provides a review of the method of singularities as a basis for generating the singularities essential for the solution of more complex problems. Solutions for some singularities for the case of transversely isotropic media are provided.

Chapter 4 presents solutions of previously-solved half-space problems for a transversely isotropic region.

Chapter 5 discusses the procedure and relationships that permit the application of the method of singularities to the solution of the problem of a sub-surface tangential force applied in a transversely isotropic medium. Some results are then plotted for materials which exhibit transversely isotropic behavior.

Chapter 6 summarizes the developments of this study and makes recommendations for further study. The use of the method of singularities to obtain solutions to other problems is also discussed.

1.6 Notation

Symbols are defined where they first appear. The symbols most frequently used are listed below:

a	c_{11} and c_{22} in the Hookean matrix; elastic constant
$\overline{a}$	c_{33} in the Hookean matrix; elastic constant
b	c_{13} and c_{33} in the Hookean matrix; elastic constant
c_{ij}	elastic constants in the Hookean matrix
E	Young's modulus; first Young's modulus for transverse isotropy
$\overline{E}$	second Young's modulus for transverse isotropy
e_ℓ	Cartesian components of "infinitesimal strain"
$k_\ell(\ell=1,2)$	constant defined by Equation (2.3.1)
M_1^*	$[\nu_1^2 r^2 + (z - \frac{s\nu_1}{\nu_2})^2]^{\frac{1}{2}}$
m_1	$M_1^* + z - \frac{s\nu_1}{\nu_2}$
N_2^*	$[\nu_2^2 r^2 + (z - \frac{s\nu_2}{\nu_1})^2]^{\frac{1}{2}}$
n_2	$N_2^* + z - \frac{s\nu_2}{\nu_1}$
$Q_\ell^*(\ell=1,2,3)$	$[\nu_\ell^2 r^2 + (z-s)^2]^{\frac{1}{2}}$
$q_\ell(\ell=1,2,3)$	$Q_\ell^* + z-s$
R	the distance from the observation point to the point of application
$R_\ell^*(\ell=1,2,3)$	$[\nu_\ell^2 r^2 + (z+s)^2]^{\frac{1}{2}}$
r	$[x^2 + y^2]^{\frac{1}{2}}$
$r_\ell(\ell=1,2,3)$	$R_\ell^* + z + s$
s	a distance
U_1, U_2, U_3	displacements in the principal Cartesian coordinate directions associated with x,y, and z respectively
W	strain energy function
x,y,z	the principal directions in the Cartesian coordinate system

μ	c_{44} and c_{55} in the Hookean matrix; elastic constant
$\overline{\mu}$	c_{66} in the Hookean matrix; elastic constant
ν	Poisson's ratio; first Poisson's ratio for transverse isotropy
$\overline{\nu}$	second Poisson's ratio for transverse isotropy
$\nu_\ell (\ell=1,2,3)$	constant defined by Equation (2.3.1)
σ_{ij}	stress
τ	Cartesian component of shear stress
$\phi_\ell \ (\ell=1,2)$	volumetric potential function (irrotational)
ψ	deviatoric potential function (equivoluminal)

2. A DISPLACEMENT-POTENTIAL REPRESENTATION FOR TRANSVERSE ISOTROPY

2.1 General

In this chapter, a representation for the displacements of transverse isotropy in terms of three potential-functions which satisfy second order linear partial differential equations is presented. It is shown that this representation is complete for displacement fields in equilibrium. A few consequences of this representation are then discussed.

2.2 Background

The Helmholtz representation of a vector function as the sum of the gradient of a scalar and the curl of a vector function (Phillips (1933)) leads one to represent the elastic displacement field in this form and to seek to uncouple the equilibrium equations. In the isotropic case this approach leads to the Love-Galerkin representation and the Boussinesq-Papkovich-Neuber representation.

In a classic paper on anisotropy, Fredholm (1900) implies the existence of both gradient-potential and curl-potential solutions of the equilibrium equations for anisotropy, but apparently Lekhnitsky (1940) first explicity stated a displacement potential approach for transverse isotropy which is complete for a class of problems, namely those characterized by torsionless axisymmetry for the case in which the axis of elastic symmetry coincides with the axis of stress symmetry.

Elliott (1948) presented a gradient-potential solution in terms of two potential functions each of which satisfy a second-order partial differential equation. Eubanks and Sternberg (1954) demonstrated the equivalence of the Elliott and Lekhnitsky representations and proved completeness for the case of rotational symmetry.

Although Elliott's approach has been used to solve problems without rotational symmetry, Hu (1953) was apparently the first investigator to introduce a curl-potential solution to accommodate a non-rotationally-symmetric problem. Subsequently, he and other investigators, presented solutions which were based on two gradient-potential functions and a curl-potential function but there was no certainty that all solutions of the three-dimensional equations of equilibrium for transverse isotropy could be represented in that way.

In the next section, this sufficiency is shown.

2.3 Displacement-Potential Representation

Let $$k_1 k_2 = 1$$

$$k_1 + k_2 = \frac{a\bar{a} - \mu^2 - (b+\mu)^2}{\mu(b+\mu)}$$

$$\nu_1{}^2 = \frac{\bar{a}\,k_1}{b+\mu + \mu k_1} = \frac{k_1(b+\mu) + \mu}{a}$$

$$\nu_2{}^2 = \frac{}{b+\mu + \mu k_2} = \frac{k_2(b+\mu) + \mu}{a}$$

$$\nu_3^2 = \frac{\mu}{\bar{\mu}}$$

$$\nabla_i^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \nu_i^2 \frac{\partial^2}{\partial z^2} \quad (i = 1,2,3) \tag{2.3.1}$$

Then a displacement field (U_1, U_2, U_3) is a solution of the transversely isotropic equations of equilibriym in a regular region V of space if and only if it admits the following representation in terms of sufficiently smooth potential functions ϕ_1, ϕ_2, and ψ.

$$U_1 = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x} + \frac{\partial \psi}{\partial y}$$

$$U_2 = \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial y} - \frac{\partial \psi}{\partial x}$$

$$U_3 = k_1 \frac{\partial \phi_1}{\partial z} + k_2 \frac{\partial \phi_2}{\partial z} \tag{2.3.2}$$

where

$$\nabla_1^2 \phi_1 = 0$$

$$\nabla_2^2 \phi_2 = 0$$

$$\nabla_3^2 \psi = 0$$

in the region V.

The fact that the above representation assures satisfaction of the equations of equilibrium is easily verified by direct substitution into the equilibrium equations. It remains to be shown that every displacement field which is in equilibrium will admit this representation.

Let

$P(x,y,z)$ and $P'(x',y',z')$ be two boints in V and let

$$R = |\overline{pp}'| = [\ (x - x^1)^2 + (y - y')^2 + (z - z')^2]^{\frac{1}{2}}$$

Let $\phi'(x',y',z')$ be the value of the function $\phi(x,y,z)$ at the point P'. Given a displacement field (U_1, U_2, U_3) in equilibrium we define the displacement potentials:

$$\phi_1 = \frac{1}{4\pi} \frac{1-\nu_1^2}{\nu_1^2 - k_1^2\nu_2^2} \int_V [\frac{\partial U_1'}{\partial x'} + \frac{\partial U_2'}{\partial y'} + k_1\ \nu_2^2 \frac{\partial U_3'}{\partial z'}] \frac{dV'}{R}$$

$$\phi_2 = \frac{1}{4\pi} \frac{1-\nu_2^2}{\nu_2^2 - k_2^2\nu_1^2} \int_V [\frac{\partial U_1'}{\partial x'} + \frac{\partial U_2'}{\partial y'} + k_2\ \nu_1^2 \frac{\partial U_3'}{\partial z'}] \frac{dV'}{R}$$

$$\psi = \frac{1}{4\pi} \frac{1-\nu_2^3}{\nu_3^2} \int_V [\frac{\partial U_1'}{\partial y'} - \frac{\partial U_2'}{\partial x'}] \frac{dV'}{R} \qquad (2.3.3)$$

Utilization of the relation:

$$\phi = \frac{1}{4\pi} \int_V [\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi'}{\partial z'^2}] \frac{dV'}{R} \qquad (2.3.4)$$

permits one to verify that these functions do, indeed reproduce U_1, U_2, and U_3, and that the defined functions satisfy the specified differential equations as a result of the equilibrium of U_1', U_2', U_3'.

Although these representations formally break down when $\nu_1{}^2 = \nu_2{}^2$, the solutions which are obtained for the general case where $\nu_1{}^2 = \nu_2{}^2$ assume correct limiting forms.

2.4 Resultant Forces Expressed in Terms of Potential Functions.

The determination of the resultant force associated with singular solutions can be tedious for complicated stress fields. This task is greatly simplified if the potentials which generate the stress field are known. In this section we demonstrate the procedure for the determination of the z-direction and x-direction resultants of the tractions on the interior of a hemispherical surface which is centered at the origin and which lies in the lower half-space.

Let $\overline{R} = (x,y,z)$ be the position vector in space and consider a hemispherical surface S: $|\overline{R}| = \alpha$, $z < 0$, with boundary c: $|\overline{R}| = \alpha$, $z = 0$. For the interior surface the normal vector is $\overline{n} = -\dfrac{\overline{R}}{\alpha}$. On the curve C we have

$$\begin{aligned} x &= \alpha \cos \theta \\ y &= \alpha \sin \theta \end{aligned} \qquad 0 \leq \theta \leq 2\pi$$

so that the arc distance is given by $s = \alpha\,\theta$ Stokes Theorem (Phillips, 1933) states that for a sufficiently smooth vector field $\overline{F}$ we have

$$\int_S \overline{n}.\ \text{curl}\ \overline{F}\ ds = \oint_C \overline{F}.\ \frac{d\overline{R}}{ds}\ ds \qquad (2.4.1.)$$

We can use this relationship to reduce the order of the integration required to determine a resultant force on S in a given direction provided the traction vector in that direction can be expressed as the curl of a vector. When the prescribed direction is x, y, or z the equilibrium equations require that the traction vector by solenoidal, hence the existance of a formulation as the curl of a vector field is assured.

z-resultant, gradient solution:

We have

$$\tau_{13} = \mu(k_1+1)\frac{\partial^2\phi_1}{\partial x\partial z} + \mu(k_2+1)\frac{\partial^2\phi_2}{\partial x\partial z}$$

$$\tau_{23} = \mu(k_1+1)\frac{\partial^2\phi_1}{\partial y\partial z} + \mu(k_2+1)\frac{\partial^2\phi_2}{\partial y\partial z}$$

$$\tau_{33} = \nu_1{}^2\mu(k_1+1)\frac{\partial^2\phi_1}{\partial z^2} + \nu_2{}^2\mu(k_2+1)\frac{\partial^2\phi_2}{\partial z^2} \qquad (2.4.2)$$

and

$$(\tau_{13}, \tau_{23}, \tau_{33}) = \text{curl}\,\overline{F} \qquad (2.4.3)$$

provided that

$$F_1 = \mu(k_1+1)\frac{\partial\phi_1}{\partial x} + \mu(k_2+1)\frac{\partial\phi_2}{\partial y}$$

$$F_2 = -\mu(k_1+1)\frac{\partial\phi_1}{\partial x} - \mu(k_2+1)\frac{\partial\phi_2}{\partial y}$$

$$F_3 = 0 \qquad (2.4.4)$$

(Note that this determination is not unique.)

The force resultant in the z-direction is

$$Z = \int_S \overline{n} \cdot (\tau_{13}, \tau_{23}, \tau_{33})\, da$$

$$= \alpha \int_0^{2\pi} \overline{F} \cdot (-\,\overline{i} \sin\theta + \overline{j} \cos\theta)\, d\theta \qquad (2.4.5)$$

where $\overline{i}$ and $\overline{j}$ are unit vectors in the x-and y-directions.

In general cylindrical coordinates we have

$$x = \rho\cos\theta$$

$$y = \rho\sin\theta$$

$$z = z$$

$$\rho^2 = x^2 + y^2 \qquad (2.4.6)$$

so that

$$-F_1 \sin\theta + F_2\cos\theta = -\mu(k_1+1)\frac{\partial\phi_1}{\partial\rho} - \mu(k_2+1)\frac{\partial\phi_2}{\partial\rho}$$

and the resultant force can be written

$$Z = -\mu\,\alpha(k_1+1)\int_0^{2\pi}\frac{\partial\phi_1}{\partial\rho}\,d\theta - \mu\,\alpha(k_2+1)\int_0^{2\pi}\frac{\partial\phi_2}{\partial\rho}\,d\theta \qquad (2.4.7)$$

where $\frac{\partial\phi_1}{\partial\rho}$ and $\frac{\partial\phi_2}{\partial\rho}$ are evaluated on C, i.e. for $\rho = \alpha$, $z = 0$.

An important special case arises for rotationally symmetric torsionless fields, where ϕ_1 and ϕ_2 are independent of θ.

in this case we have

$$Z = -2\pi \mu \alpha \left[(k_1+1)\frac{\partial \phi_1}{\partial \rho} + (k_2+1)\frac{\partial \phi_2}{\partial \rho}\right]_{\substack{z=0\\ \rho=\alpha}} \tag{2.4.8}$$

z resultant, curl solution

The stresses are given by

$$\tau_{13} = \mu \frac{\partial^2 \psi}{\partial y \partial z}$$

$$\tau_{23} = -\mu \frac{\partial^2 \psi}{\partial x \partial z}$$

$$\tau_{33} = 0 \tag{2.4.9}$$

and we find that

$$(\tau_{13}, \tau_{23}, \tau_{33}) = \text{curl } \overline{F} \tag{2.4.10}$$

if

$$F_1 = F_2 = 0, \; F_3 = \mu \frac{\partial \psi}{\partial z} \tag{2.4.11}$$

thus, the z-force is

$$Z = \alpha \int_0^{2\pi} \overline{F} \cdot (-\overline{i} \sin\theta + \overline{j} \cos\theta)\, d\theta = 0$$

x resultant - gradient solution:

$$\tau_{11} = -\mu(k_1+1)\frac{\partial^2 \phi_1}{\partial z^2} - \mu(k_2+1)\frac{\partial^2 \phi_2}{\partial z^2} - 2\overline{\mu}\left(\frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial y^2}\right)$$

$$\tau_{12} = 2\overline{\mu}\left(\frac{\partial^2 \phi_1}{\partial x \partial y} + \frac{\partial^2 \phi_2}{\partial x \partial y}\right)$$

$$\tau_{13} = \mu(k_1+1)\,\frac{\partial^2\phi_1}{\partial x\partial z} + \mu(k_2+1)\,\frac{\partial^2\phi_2}{\partial x\partial z} \tag{2.4.12}$$

and

$$(\tau_{11}, \tau_{12}, \tau_{13}) = \text{curl}\ \overline{F} \tag{2.4.13}$$

provided

$$F_1 = 0$$

$$F_2 = \mu(k_1+1)\,\frac{\partial\phi_1}{\partial z} + \mu(k_2+1)\,\frac{\partial\phi_2}{\partial z}$$

$$F_3 = -2\bar{\mu}\left(\frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial y}\right) \tag{2.4.14}$$

The force in the x-direction is

$$X = \int_{S_2} \overline{n}\cdot(\tau_{11}\ \ \tau_{12}\ \ \tau_{13})\,da$$

$$= \alpha\int_0^{2\pi}\left[\,(k_1+1)\,\frac{\partial\phi_1}{\partial z} + \mu(k_2+1)\,\frac{\partial\phi_2}{\partial z}\,\right]\cos\theta\ \ d\theta \qquad 2.4.15)$$

<u>x-resultant -curl solution:</u>

The stresses are given by

$$\tau_{11} = 2\bar{\mu}\ \frac{\partial^2\psi}{\partial x\partial y}$$

$$\tau_{12} = \bar{\mu}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) = -\mu\,\frac{\partial^2\psi}{\partial z^2} - 2\bar{\mu}\,\frac{\partial^2\psi}{\partial x^2}$$

$$\tau_{13} = \mu\,\frac{\partial^2\psi}{\partial y\partial z}$$

and

$$(\tau_{11}, \tau_{12}, \tau_{13}) = \mathrm{curl}\, \overline{F}$$

provided that

$$F_1 = -\mu \frac{\partial\psi}{\partial z}$$

$$F_2 = 0$$

$$F_3 = 2\overline{\mu} \frac{\partial\psi}{\partial x}$$

Thus, the x-force is

$$X = \int_{S_2} \overline{n} \cdot (\tau_{11}, \tau_{12}, \tau_{13})\, da$$

$$= \alpha \int_0^{2\pi} \mu \frac{\partial\psi}{\partial z} \sin\theta\, d\theta$$

Other required resultants can be computed with similar techniques.

3. BASIC SINGULARITIES

3.1 General

This chapter considers the various singularities in a transversely isotropic medium which are the necessary building-blocks in the method of singularities. As mentioned in Section 1.2, many higher and lower order singularities can be generated for isotropic media. Similar approaches hold for transverse isotropy. This section will present only those singularities which are of specific interest to the solution of the problems in Chapters 4 and 5.

The three-dimensional Cartesian coordinate system will be utilized with the principal directions denoted by x, y and z. The z-direction will always be considered the vertical direction, and the medium will always have symmetry with respect to the z-axis.

3.2 Horizontal Unit-Force in the x-Direction

Love (1926), Kröner (1953), Hu (1954), Woo and Shield (1962), and, Pan and Chou (1976) have used this solution for the case of transverse isotropy in their work. The form presented here differs slightly from their solutions but the stresses and displacements are the same.

Letting $R_\ell = \nu_\ell^2 r^2 + z^2$ (ℓ= 1,2,3), the potential functions are (see Figure 3.1)

$$\phi_1 = \frac{1}{4\pi\mu(k_1^2-1)} \left\{ \frac{xz}{r^2} - \frac{xR_1}{r^2} \right\} = \frac{1}{4\pi\mu(k_1^2-1)} \left\{ \frac{-\nu_1^2 x}{R_1+z} \right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1^2-1)} \left\{ \frac{-k_1^2 xz}{r^2} + \frac{k_1^2 xR_2}{r^2} \right\} = \frac{1}{4\pi\mu(k_1^2-1)} \left\{ \frac{k_1^2 \nu_2^2 x}{R_2 + z} \right\}$$

$$\psi = \frac{1}{4\pi\mu} \left\{ - \frac{yz}{r^2} + \frac{yR_3}{r^2} \right\} = \frac{1}{4\pi\mu} \left\{ \frac{\nu_3^2 y}{R_3+z} \right\} \tag{3.2.1}$$

The displacements and stresses are:

$$U_1 = \frac{1}{4\pi\mu(k_1^2-1)} \left\{ \frac{\nu_1^4 x^2}{R_1(R_2+z)^2} - \frac{\nu_1^2}{R_1+z} \right\} + \frac{1}{4\pi\mu(k_1^2-1)} \left\{ \frac{-k_1^2\nu_2^4 x^2}{R_2(R_2+z)^2} + \frac{k_1^2\nu_2^2}{R_2+z} \right\}$$

$$+ \frac{1}{4\pi\mu} \left\{ \frac{-\nu_3^4 y^2}{R_3(R_3+z)^2} + \frac{\nu_3^2}{R_3+z} \right\} \tag{3.2.2}$$

$$U_2 = \frac{1}{4\pi\mu(k_1^2-1)} \left\{ \frac{\nu_1^4 xy}{R_1(R_1+z)^2} \right\} + \frac{1}{4\pi\mu(k_1^2-1)} \left\{ \frac{-k_1^2\nu_2^4 xy}{R_2(R_2+z)^2} \right\}$$

$$+ \frac{1}{4\pi\mu} \left\{ \frac{\nu_3^4 xy}{R_3(R_3+z)^2} \right\} \tag{3.2.3}$$

$$U_3 = \frac{1}{4\pi\mu(k_1{}^2-1)}\left\{\frac{k_1\nu_1{}^2 x}{R_1(R_1+z)}\right\} + \frac{1}{4\pi\mu(k_1{}^2-1)}\left\{\frac{-k_1\nu_2{}^2 x}{R_2(R_2+z)}\right\} \tag{3.2.4}$$

$$\sigma_{xz} = \frac{1}{4\pi(k_1-1)}\left\{\frac{-2\nu_1{}^4y^2z}{R_1(R_1{}^2-z^2)^2} + \frac{\nu_3{}^2z}{R_1(R_1{}^2-z^2)} + \frac{\nu_1{}^4x^2z}{R_1{}^3(R_1{}^2-z^2)}\right\}$$

$$+ \frac{1}{4\pi(k_1-1)}\left\{+\frac{2k_1\nu_2{}^4y^2z}{R_2(R_2{}^2-z^2)^2} - \frac{k_1\nu_2{}^2z}{R_2(R_2{}^2-z^2)} - \frac{k_1\nu_1{}^4x^2z}{R_2{}^3(R_2{}^2-z^2)}\right\}$$

$$+ \frac{1}{4\pi}\left\{-\frac{2\nu_3{}^4y^2z}{R_3(R_3{}^2-z^2)} + \frac{\nu_3{}^2z}{R_3(R_3{}^2-z^2)} - \frac{\nu_3{}^4y^2z}{R_3{}^3(R_3{}^2-z^2)}\right\} \tag{3.2.5}$$

$$\sigma_{yz} = \frac{1}{4\pi(k_1-1)}\left\{\frac{2\nu_1{}^4xyz}{R_1(R_1{}^2-z^2)} + \frac{\nu_1{}^4xyz}{R_1{}^3(R_1{}^2-z^2)}\right\}$$

$$+ \frac{1}{4\pi(k_1-1)}\left\{-\frac{2k_1\nu_2{}^4xyz}{R_2(R_2{}^2-z^2)^2} - \frac{k_1\nu_2{}^4xyz}{R_2{}^3(R_2{}^2-z^2)}\right\}$$

$$+ \frac{1}{4\pi}\left\{\frac{2\nu_3{}^4xyz}{R_3(R_3{}^2-z^2)^2} + \frac{\nu_3{}^4xyz}{R_3{}^3(R_3{}^2-z^2)}\right\} \tag{3.2.6}$$

$$\sigma_{zz} = \frac{1}{4\pi(k_1-1)}\left\{-\frac{\nu_1{}^4x}{R_1{}^3}\right\} + \frac{1}{4\pi(k_1-1)}\left\{\frac{k_1\nu_2{}^4x}{R_2{}^3}\right\} * \tag{3.2.7}$$

* Woo and Shield's (1962) evaluation of σ_{zz} at z=0 contains a typographical error.

$$\sigma_{xx} = \frac{1}{2\pi\nu_3^{\,2}(k_1^{\,2}-1)}\{-\frac{\nu_1^{\,6}xy^2z^4}{R_1^{\,3}(R_1^{\,2}-z^2)^3} - \frac{\nu_1^{\,4}xR_1}{(R_1^{\,2}-z^2)^2} + \frac{3\nu_1^{\,6}xy^2R_1}{(R_1^{\,2}-z^2)^3}$$

$$+ \frac{5\nu_1^{\,6}xy^2z^2}{R_1(R_1^{\,2}-z^2)^3} + \frac{\nu_1^{\,2}\nu_3^{\,2}x(k_1+1)}{2R_1^{\,3}}$$

$$+ \frac{1}{2\pi\nu_3^{\,2}(k_1^{\,2}-1)}\{+\frac{k_1^{\,2}\nu_2^{\,6}xy^2z^4}{R_2^{\,3}(R_2^{\,2}-z^2)^3} + \frac{k_1^{\,2}\nu_2^{\,4}xR_2}{(R_2^{\,2}-z^2)^2} - \frac{3k_1^{\,2}\nu_2^{\,6}xy^2R_2}{(R_2^{\,2}-z^2)^3}$$

$$- \frac{5k_1^{\,2}\nu_2^{\,6}xy^2z^2}{R_2(R_2^{\,2}-z^2)^3} - \frac{k_1\nu_2^{\,2}\nu_3^{\,2}x(k_1+1)}{2R_2^{\,3}}$$

$$+ \frac{1}{2\pi}\{-\frac{\nu_3^{\,4}xy^2z^4}{R_3^{\,3}(R_3^{\,2}-z^2)^3} - \frac{\nu_3^{\,2}xR_3}{(R_3^{\,2}-z^2)^2} + \frac{3\nu_3^{\,4}xy^2R_3}{(R_3^{\,2}-Z^2)^3} + \frac{5\nu_3^{\,4}xy^2z^2}{R_3(R_3^{\,2}-z^2)^3} \tag{3.2.8}$$

$$\sigma_{yy} = \frac{1}{2\pi\nu_3^{\,2}(k_1^{\,2}-1)}\{-\frac{3\nu_1^{\,6}xy^2R_1}{(R_1^{\,2}-z^2)^3} - \frac{\nu_1^{\,6}x^3z^4}{R_1^{\,3}(R_1^{\,2}-z^2)^3} + \frac{3\nu_1^{\,6}xz^2(x^2-y^2)}{R_1(R_1^{\,2}-z^2)^3}$$

$$+ \frac{\nu_1^{\,2}\nu_3^{\,2}x(k_1+1)}{2R_1^{\,3}}\} + \frac{1}{2\pi\nu_3^{\,2}(k_1^{\,2}-1)}\{+\frac{3k_1^{\,2}\nu_2^{\,6}xy^2R_2}{(R_2^{\,2}-z^2)^3} + \frac{k_1^{\,2}\nu_2^{\,6}x^3z^4}{R_2^{\,3}(R_2^{\,2}-z^2)^3}$$

$$- \frac{3k_1^{\,2}\nu_2^{\,6}xz^2(x^2-y^2)}{R_2(R_2^{\,2}-z^2)^3} - \frac{k_1\nu_2^{\,2}\nu_3^{\,2}x(k_1+1)}{2R_2^{\,3}}\} + \frac{1}{2\pi}\{\frac{3\nu_3^{\,4}xy^2R_3}{(R_3^{\,2}-z^2)^3}\}$$

$$+ \frac{\nu_3^{\,4}x^3z^4}{R_3^{\,3}(R_3^{\,2}-z^2)^3} - \frac{3\nu_3^{\,4}xz^2(x^2-y^2)}{R_3(R_3^{\,2}-z^2)^3} \tag{3.2.9}$$

$$\sigma_{xy} = \frac{1}{2\pi\nu_3^2(k_1^2-1)}\{-\frac{8\nu_1^6x^2yR_1}{(R_1^2-z^2)^3} - \frac{\nu_1^4y(R_1^2-\nu_1^2x^2)}{R_1^3(R_1^2-z^2)} + \frac{2\nu_1^4y(2\nu_1^2x^2+R_1^2)}{R_1(R_1^2-z^2)^2}\}$$

$$+ \frac{1}{2\pi\nu_3^2(k_1^2-1)}\{-\frac{k_1\nu_2^6x^2y}{R_2^3(R_2^2-z^2)} + \frac{8k_1^2\nu_2^6x^2yR_2}{(R_2^2-z^2)^3}$$

$$+ \frac{k_1^2\nu_2^4y(\nu_2^2y^2-3\nu_2^2x^2-2R_2^2}{R_2(R_2^2-z^2)^2}\}$$

$$+ \frac{1}{4\pi}\{-\frac{\nu_3^2y}{R_3^3} + \frac{\nu_3^2y(2\nu_3^2x^2+5R_3^2)}{R_3^3(R_3^2-z^2)} - \frac{7\nu_3^4y^3}{R_3(R_3^2-z^2)^2}\} \qquad (3.2.10)$$

3.3 Vertical Unit-Force in the z-Direction

See Figure 3.2, we have

$$\phi_1 = \frac{1}{8\pi\mu(k_1^2-1)}\{k_1 \log\left(\frac{R_1+z}{R_1-z}\right) + 2k_1 \log r\}$$

$$= \frac{1}{4\pi\mu(k_1^2-1)}\{-k_1 \log(R_1+z)\}$$

$$\phi_2 = \frac{1}{8\pi\mu(k_1^2-1)}\{-k_1 \log\left(\frac{R_2+z}{R_2-z}\right) - 2k_1 \log r\}$$

$$= \frac{1}{4\pi\mu(k_1^2-1)}\{-k_1 \log(R_2+z)\}$$

$$\psi = 0$$

The displacements and stresses become:

$$U_1 = \frac{1}{4\pi\mu(k_1{}^2-1)}\{-\frac{k_1\nu_1{}^2xz}{R_1(R_1{}^2-z^2)}\} + \frac{1}{4\pi\mu(k_1{}^2-1)}\{\frac{k_1\nu_2{}^2xz}{R_2(R_2{}^2-z^2)}\} \quad (3.3.2)$$

$$U_2 = \frac{1}{4\pi\mu(k_1{}^2-1)}\{-\frac{k_1\nu_1{}^2yz}{R_1(R_1{}^2-z^2)}\} + \frac{1}{4\pi\mu(k_1{}^2-1)}\{\frac{k_1\nu_2{}^2yz}{R_2(R_2{}^2-z^2)}\} \quad (3.3.3)$$

$$U_3 = \frac{1}{4\pi\mu(k_1{}^2-1)}\{\frac{k_1{}^2}{R_1}\} + \frac{1}{4\pi\mu(k_1{}^2-1)}\{-\frac{1}{R_2}\} \quad (3.3.4)$$

$$\sigma_{xz} = \frac{1}{4\pi(k_1-1)}\{-\frac{k_1\nu_1{}^2x}{R_1{}^3}\} + \frac{1}{4\pi(k_1\ -1)}\{\frac{\nu_2{}^2x}{R_2{}^3}\} \quad (3.3.5)$$

$$\sigma_{yz} = \frac{1}{4\pi(k_1-1)}\{-\frac{k_1\nu_1{}^2y}{R_1{}^3}\} + \frac{1}{4\pi(k_1-1)}\{\frac{\nu_2{}^2y}{R_2{}^3}\} \quad (3.3.6)$$

$$\sigma_{zz} = \frac{1}{4\pi(k_1-1)}\{-\frac{k_1\nu_1{}^2z}{R_1{}^3}\} + \frac{1}{4\pi(k_1-1)}\{\frac{\nu_2{}^2z}{R_2{}^3}\} \quad (3.3.7)$$

$$\sigma_{xx} = \frac{1}{4\pi\mu(k_1{}^2-1)}\{\frac{ak_1\nu_1{}^2z}{R_1(R_1{}^2-z^2)} + \frac{2ak_1\nu_1{}^4yz(x-y)}{R_1(R_1{}^2-z^2)^2} - \frac{ak_1\nu_1{}^4yz(x-y)}{R_1{}^3(R_1{}^2-z^2)}$$

$$+ \frac{ak_1\nu_1{}^2z}{R_1{}^3} - \frac{bk_1{}^2z}{R_1{}^3} - \frac{4\bar{\mu}k_1\nu_1{}^4xyz}{R_1(R_1{}^2-z^2)^2} - \frac{2\bar{\mu}k_1\nu_1{}^4xyz}{R_1{}^3(R_1{}^2-z^2)}\}$$

$$+ \frac{1}{4\pi\mu(k_1{}^2-1)}\{-\frac{ak_1\nu_2{}^2z}{R_2(R_2{}^2-z^2)} - \frac{2ak_1\nu_2{}^4yz(x-y)}{R_2(R_2{}^2-z^2)^2} + \frac{ak_1\nu_2{}^4yz(x-y)}{R_2{}^3(R_2{}^2-z^2)}$$

$$- \frac{ak_1\nu_2{}^2z}{R_2{}^3} + \frac{bz}{R_2{}^3} + \frac{4\bar{\mu}k_1\nu_2{}^4xyz}{R_2(R_2{}^2-z^2)^2} + \frac{2\bar{\mu}k_1\nu_2{}^4xyz}{R_2{}^3(R_2{}^2-z^2)}\} \quad (3.3.8)$$

$$\sigma_{yy} = \frac{1}{4\pi\mu(k_1{}^2-1)} \{ \frac{ak_1\nu_1{}^2 z}{R_1(R_1{}^2-z^2)} + \frac{2ak_1\nu_1{}^4 yz(x-y)}{R_1(R_1{}^2-z^2)^2} - \frac{ak_1\nu_1{}^4 yz(x-y)}{R_1{}^3(R_1{}^2-z^2)}$$

$$+ \frac{ak_1\nu_1{}^2 z}{R_1{}^3} - \frac{bk_1{}^2 z}{R_1{}^3} + \frac{4\bar{\mu}k_1\nu_1{}^4 y^2 z}{R_1(R_1{}^2-z^2)^2} + \frac{2\bar{\mu}k_1\nu_1{}^4 y^2 z}{R_1{}^3(R_1{}^2-z^2)} \}$$

$$+ \frac{1}{4\pi\mu(k_1{}^2-1)} \{ - \frac{ak_1\nu_2{}^2 z}{R_2(R_2{}^2-z^2)} - \frac{2ak_1\nu_2{}^4 yz(x-y)}{R_2(R_2{}^2-z^2)^2} + \frac{ak_1\nu_2{}^4 yz(x-y)}{R_2{}^3(R_2{}^2-z^2)}$$

$$- \frac{ak_1\nu_2{}^2 z}{R_2{}^3} + \frac{bz}{R_2{}^3} - \frac{4\bar{\mu}k_1\nu_2{}^4 y^2 z}{R_2(R_2{}^2-z^2)^2} - \frac{2\bar{\mu}k_1\nu_2{}^4 y^2 z}{R_2{}^3(R_2{}^2-z^2)} \qquad (3.3.9)$$

$$\sigma_{xy} = \frac{1}{2\pi\nu_3{}^2(k_1{}^2-1)} \{ \frac{2k_1\nu_1{}^4 xyz}{R_1(R_1{}^2-z^2)^2} + \frac{k_1\nu_1{}^4 xyz}{R_1{}^3(R_1{}^2-z^2)} \}$$

$$+ \frac{1}{2\pi\nu_3{}^2(k_1{}^2-1)} \{ - \frac{2k_1\nu_2{}^4 xyz}{R_2(R_2{}^2-z^2)^2} - \frac{k_1\nu_2{}^4 xyz}{R_2{}^3(R_2{}^2-z^2)} \} \qquad (3.3.10)$$

3.4 Double Force Without Moment

The derivative of the Kelvin solution in the direction of its force generates what Love (1926) called a "double force without moment". Computing this force doublet from the forcd in the x-direction yields the potentials (see Figure 3.3(a)):

$$\phi_1 = \frac{1}{4\pi\mu(k_1{}^2-1)} \{ \frac{x^2 z^2}{r^4 R_1} - \frac{y^2 R_1}{r^4} + \frac{z}{r^2} - \frac{2x^2 z}{r^4} \}$$

or,

$$\phi_1 = \frac{1}{4\pi\mu(k_1{}^2-1)} \left\{ \frac{\nu_1{}^4 x^2}{R_1(R_1+z)^2} - \nu_1{}^2 \right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1{}^2-1)} \left\{ - \frac{k_1{}^2 x^2 z^2}{r^4 R_2} + \frac{k_1{}^2 y^2 R_2}{r^4} - \frac{k_1{}^2 z}{r^2} + \frac{2k_1{}^2 x^2 z}{r^4} \right\}$$

$$= \frac{1}{4\pi\mu(k_1{}^2-1)} \left\{ - \frac{k_1{}^2 \nu_2{}^4 x^2}{R_2(R_2+z)^2} + k_1{}^2 \nu_2{}^2 \right\}$$

$$\psi = \frac{1}{4\pi\mu} \left\{ - \frac{xyz^2}{r^4 R_3} - \frac{xyR_3}{r^4} + \frac{2xyz}{r^4} \right\}$$

$$= \frac{1}{4\pi\mu} \left\{ - \frac{\nu_3{}^4 xy}{R_3(R_3+z)^2} \right\} \qquad (3.4.1)$$

Similarly (see Figure 3.3(b)) for a force doublet in the z-direction the potentials are:

$$\phi_1 = \frac{1}{4\pi\mu(k_1{}^2-1)} \left\{ \frac{k_1}{R_1} \right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1{}^2-1)} \left\{ - \frac{k_1}{R_2} \right\}$$

$$\psi = 0$$

This singularity is, of course, axially symmetrical about the z-axis and therefore, and as expected, the deviatoric potential ψ is equal to zero.

The double force without moment is not explicitly used in the solution of problems in Chapters 4 and 5. Therefore, the stresses are not presented here.

3.5 Center of Dilatation

Superposing three "double forces without moment" which are mutually orthogonal, creates a "center of dilatation" or a "center of compression" depending on the orientation of the forces. Therefore a "center of dilatation" assumes the form (see Figure 3.4):

$$\phi_1 = \frac{1}{4\pi\mu(k_1{}^2-1)} \left\{ \frac{k_1-\nu_1{}^2}{R_1} \right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1{}^2-1)} \left\{ \frac{k_1{}^2\nu_2{}^2-k_1}{R_2} \right\}$$

$$\psi = 0$$

The center of dilatation is not explicitly used in the solution of problems in Chapters 4 and 5 and, therefore the displacements and stresses are omitted.

3.6 Line of Dilatation

Line singularities can also be generated in addition to those which are singular at a point. One such line may be, say, one which extends along the z-axis from the origin to infinity. Integrating a center of dilatation along the z-axis from zero to infinity produces the potential functions:

$$\phi_1 = \frac{1}{2\pi\mu(k_1{}^2-1)} \{ (k_1-\nu_1{}^2) \log (R_1+z)\}$$

$$\phi_2 = \frac{1}{2\pi\mu(k_1{}^2-1)} \{ (k_1{}^2\nu_2{}^2-k_1) \log (R_2+z) \}$$

$$\psi = 0 \tag{3.6.1}$$

The related displacements and stresses are:

$$U_1 = \frac{1}{2\pi\mu(k_1{}^2-1)} \{ \frac{\nu_1{}^2x(k_1-\nu_1{}^2)}{R_1(R_1+z)} \} + \frac{1}{2\pi\mu(k_1{}^2-1)} \{ \frac{k_1\nu_2{}^2x(k_1\nu_2{}^2-1)}{R_2(R_2+z)} \} \tag{3.6.2}$$

$$U_2 = \frac{1}{2\pi\mu(k_1{}^2-1)} \{ \frac{\nu_1{}^2y(k_1-\nu_1{}^2)}{R_1(R_1+z)} \} + \frac{1}{2\pi\mu(k_1{}^2-1)} \{ \frac{k_1\nu_2{}^2y(k_1\nu_2{}^2-1)}{R_2(R_2+z)} \tag{3.6.3}$$

$$U_3 = \frac{1}{2\pi\mu(k_1{}^2-1)} \{ \frac{k_1(k_1-\nu_1{}^2)}{R_1} \} + \frac{1}{2\pi\mu(k_1{}^2-1)} \{ \frac{(k_1\nu_2{}^2-1)}{R_2} \} \tag{3.6.4}$$

$$\sigma_{xz} = \frac{1}{2\pi(k_1-1)} \{- \frac{\nu_1{}^2x(k_1-\nu_1{}^2)}{R_1{}^3} \} + \frac{1}{2\pi(k_1-1)} \{- \frac{\nu_2{}^2x(k_1\nu_2{}^2-1)}{R_2{}^3} \} \tag{3.6.5}$$

$$\sigma_{yz} = \frac{1}{2\pi(k_1-1)} \{- \frac{\nu_1{}^2y(k_1-\nu_1{}^2)}{R_1{}^3} \} + \frac{1}{2\pi(k_1-1)} \{- \frac{\nu_2{}^2y(k_1\nu_2{}^2-1)}{R_2{}^3} \} \tag{3.6.6}$$

$$\sigma_{zz} = \frac{1}{2\pi(k_1-1)} \{- \frac{\nu_1{}^2z(k_1-\nu_1{}^2)}{R_1{}^3} \} + \frac{1}{2\pi(k_1-1)} \{- \frac{\nu_2{}^2z(k_1\nu_2{}^2-1)}{R_2{}^3} \} \tag{3.6.7}$$

$$\sigma_{xx} = \frac{1}{2\pi\mu(k_1{}^2-1)} \{(k_1-\nu_1{}^2)(\frac{2a\nu_1{}^2}{R_1(R_1+z)} - \frac{2a\nu_1{}^2(R_1-z)}{R_1{}^2(R_1+z)} - \frac{a\nu_1{}^2z(R_1-z)}{R_1{}^3(R_1+z)}$$

$$- \frac{bk_1z}{R_1{}^3} - \frac{2\bar{\mu}\nu_1{}^2}{R_1(R_1+z)} + \frac{4\bar{\mu}\nu_1{}^4y^2}{R_1{}^2(R_1+z)^2} + \frac{2\bar{\mu}\nu_1{}^4y^2z}{R_1{}^3(R_1+z)^2})\}$$

$$+ \frac{1}{2\pi\mu(k_1{}^2-1)} \{(k_1\nu_2{}^2-1)(\frac{2ak_1\nu_2{}^2}{R_2(R_2+z)} - \frac{2ak_1\nu_2{}^2(R_2-z)}{R_2{}^2(R_2+z)} - \frac{ak_1\nu_2{}^2z(R_2-z)}{R_2{}^3(R_2+z)}$$

$$- \frac{bz}{R_2{}^3} - \frac{2\bar{\mu}k_1\nu_2{}^2}{R_2(R_2+z)} + \frac{4\bar{\mu}k_1\nu_2{}^4y^2}{R_2{}^2(R_2+z)^2} + \frac{2\bar{\mu}k_1\nu_2{}^4y^2z}{R_2{}^3(R_2+z)})\} \qquad (3.6.8)$$

$$\sigma_{yy} = \frac{1}{2\pi\mu(k_1{}^2-1)} \{(k_1-\nu_1{}^2)(\frac{2a\nu_1{}^2}{R_1(R_1+z)} - \frac{2a\nu_1{}^2(R_1-z)}{R_1{}^2(R_1+z)} - \frac{a\nu_1{}^2z(R_1-z)}{R_1{}^3(R_1+z)})\}$$

$$- \frac{bk_1z}{R_1{}^3} - \frac{2\bar{\mu}\nu_1{}^2}{R_1(R_1+z)} + \frac{4\bar{\mu}\nu_1{}^4x^2}{R_1{}^2(R_1+z)^2} + \frac{2\bar{\mu}\nu_1{}^4x^2z}{R_1{}^3(R_1+z)^2})\}$$

$$+ \frac{1}{2\pi\mu(k_1{}^2-1)} \{(k_1\nu_2{}^2-1)(\frac{2ak_1\nu_2{}^2}{R_2(R_2+z)} - \frac{2ak_1\nu_2{}^2(R_2-z)}{R_2{}^2(R_2+z)} - \frac{ak_1\nu_2{}^2z(R_2-z)}{R_2{}^3(R_2+z)}$$

$$- \frac{bz}{R_2{}^3} - \frac{2\bar{\mu}k_1\nu_2{}^2}{R_2(R_2+z)} + \frac{4\bar{\mu}k_1\nu_2{}^4x^2}{R_2{}^2(R_2+z)^2} + \frac{2\bar{\mu}k_1\nu_2{}^4x^2z}{R_2{}^3(R_2+z)^2})\} \qquad (3.6.9)$$

$$\sigma_{xy} = \frac{1}{\pi\nu_3{}^2(k_1{}^2-1)} \{(k_1-\nu_1{}^2)(- \frac{2\nu_1{}^4xy}{R_1{}^2(R_1+z)^2} - \frac{\nu_1{}^4xyz}{R_1{}^3(R_1+z)^2})\}$$

$$+ \frac{1}{\pi\nu_3{}^2(k_1{}^2-1)} \{(k_1\nu_2{}^2-1)(- \frac{2k_1\nu_2{}^4xy}{R_2{}^2(R_2+z)^2} - \frac{k_1\nu_2{}^4xyz}{R_2{}^3(R_2+z)^2} \} \qquad (3.6.10)$$

3.7 Double Force With Moment

The derivative of the Kelvin solution perpendicular to the direction of its force, yields what Love (1926) referred to as a "double force with moment". Taking the derivative of the x-direction force with respect to the z-direction, yields:

$$\phi_1 = \frac{1}{4\pi\mu(k_1^2-1)}\left\{\frac{x}{r^2} - \frac{xz}{r^2R_1}\right\} = \frac{1}{4\pi\mu(k_1^2-1)}\left\{\frac{\nu_1^2 x}{R_1(R_1+z)}\right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1^2-1)}\left\{-\frac{k_1^2 x}{r^2} + \frac{k_1^2 xz}{r^2R_2}\right\} = \frac{1}{4\pi\mu(k_1^2-1)}\left\{-\frac{k_1^2\nu_2^2 x}{R_2(R_2+z)}\right\}$$

$$\psi = \frac{1}{4\pi\mu}\left\{-\frac{y}{r^2} + \frac{yz}{r^2R_3}\right\} = \frac{1}{4\pi\mu}\left\{-\frac{\nu_3^2 y}{R_3(R_3+z)}\right\} \qquad (3.7.1)$$

The resultant moment in this case is about the negative y-axis (see Figure 3.5(a)).

Similarly taking the derivative of the z-direction force with respect to the x-direction:

$$\phi_1 = \frac{1}{4\pi\mu(k_1^2-1)}\left\{\frac{k_1 x}{r^2} - \frac{k_1 xz}{r^2R_1}\right\} = \frac{1}{4\pi\mu(k_1^2-1)}\left\{\frac{k_1\nu_1^2 x}{R_1(R_1+z)}\right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1^2-1)}\left\{-\frac{k_1 x}{r^2} + \frac{k_1 xz}{r^2R_2}\right\} = \frac{1}{4\pi\mu(k_1^2-1)}\left\{-\frac{k_1\nu_2^2 x}{R_2(R_2+z)}\right\}$$

$$\psi = 0$$

In this case the resultant moment is generated about the positive y-axis (see Figure 3.5(b)).

Since neither of the double forces with moment are used explicitly in the solution of problems in the later chapters, the displacements and stresses will not be presented here.

In a parallel manner, other double forces with moment can be obtained whose resultant moments are about the x- or z- axis. For the purpose of this study the needed double forces with moment are those which create moments about the y-axis. This will be made clear in the solution of problems in Chapters 4 and 5.

3.8 Center of Rotation

Combining two double forces with moment which share the same axis, that is, which create moments about the same axis, results in a center of rotation. If two of these double forces share the y-axis as those which were derived in section 3.7, a center of rotation about the y-axis is formed (see Figure 3.6).

$$\phi_1 = \frac{1}{4\pi\mu(k_1+1)} \left\{ \frac{x}{r^2} - \frac{xz}{r^2 R_1} \right\} = \frac{1}{4\pi\mu(k_1+1)} \left\{ \frac{\nu_1{}^2 x}{R_1(R_1+z)} \right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1+1)} \left\{ \frac{k_1 x}{r^2} - \frac{k_1 xz}{r^2 R_2} \right\} = \frac{1}{4\pi\mu(k_1+1)} \left\{ \frac{k_1 \nu_2{}^2 x}{R_2(R_2+z)} \right\}$$

$$\psi = \frac{1}{4\pi\mu} \left\{ \frac{y}{r^2} - \frac{yz}{r^2 R_3} \right\} = \frac{1}{4\pi\mu} \left\{ \frac{\nu_3{}^2 y}{R_3(R_3+z)} \right\} \qquad (3.8.1)$$

The center of rotation is not used explicitly in the solution of problems in Chapters 4 and 5. Therefore the displacements and stresses are omitted.

Centers of rotation about either the x- or z-axis are obtained in a parallel manner.

3.9 Line of Rotation

Taking a center of rotation about the y-axis as described in the preceeding section, and integrating it along the z-axis from the origin to infinity will create a line of rotation whose rotation centers generate moments about the y-axis:

$$\phi_1 = \frac{1}{4\pi\mu(k_1+1)}\left\{\frac{xR_1}{r^2} - \frac{xz}{r^2}\right\} = \frac{1}{4\pi\mu(k_1+1)}\left\{\frac{\nu_1{}^2 x}{(R_1+z)}\right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1+1)}\left\{\frac{k_1 x R_2}{r^2} - \frac{k_1 xz}{r^2}\right\} = \frac{1}{4\pi\mu(k_1+1)}\left\{\frac{k_1\nu_2{}^2 x}{(R_2+z)}\right\}$$

$$\psi = \frac{1}{4\pi\mu}\left\{\frac{yR_3}{r^2} - \frac{yz}{r^2}\right\} = \frac{1}{4\pi\mu}\left\{\frac{\nu_3{}^2 y}{(R_3+z)}\right\} \tag{3.9.1}$$

The displacements and stresses are given by:

$$U_1 = \frac{1}{4\pi\mu(k_1+1)}\left\{\frac{\nu_1{}^2}{(R_1+z)} - \frac{\nu_1{}^4 x^2}{R_1(R_1+z)^2}\right\}$$

$$+ \frac{1}{4\pi\mu(k_1+1)}\left\{\frac{k_1\nu_2{}^2}{(R_2+z)} - \frac{k_1\nu_2{}^4 x^2}{R_2(R_2+z)^2}\right\}$$

$$+ \frac{1}{4\pi\mu}\left\{\frac{\nu_3{}^2}{(R_3+z)} - \frac{\nu_3{}^4 y^2}{R_3(R_3+z)^2}\right\} \tag{3.9.2}$$

$$U_2 = \frac{1}{4\pi\mu(k_1+1)}\left\{\frac{\nu_1{}^4xy}{R_1(R_1{}^2-z^2)} - \frac{2\nu_1{}^4xy}{(R_1+z)\ (R_1-z)}\right\}$$

$$+ \frac{1}{4\pi\mu(k_1+1)}\left\{\frac{k_1\nu_2{}^4xy}{R_2(R_2{}^2-z^2)} - \frac{2k_1\nu_2{}^4xy}{(R_2+z)^2(R_2-z)}\right.$$

$$\left.+ \frac{1}{4\pi\mu}\left\{- \frac{\nu_3{}^4xy}{R_3(R_3{}^2-z^2)} + \frac{2\nu_2{}^4xy}{(R_3+z)^2(R_2-z)}\right\}\right. \tag{3.9.3}$$

$$U_3 = \frac{1}{4\pi\mu(k_1+1)}\left\{- \frac{k_1\nu_1{}^2x}{R_1(R_1+z)}\right\} + \frac{1}{4\pi\mu(k_1+1)}\left\{- \frac{\nu_2{}^2x}{R_2(R_2+z)}\right\} \tag{3.9.4}$$

$$\sigma_{xz} = \frac{1}{4\pi}\left\{- \frac{\nu_1{}^2}{R_1(R_1+z)} + \frac{2\nu_1{}^4x^2}{R_1(R_1+z)^2(R_1-z)} - \frac{\nu_1{}^4x^2z}{R_1{}^3(R_1{}^2-z^2)}\right\}$$

$$+ \frac{1}{4\pi}\left\{- \frac{\nu_2{}^2}{R_2(R_2+z)} + \frac{2\nu_2{}^4x^2}{R_2(R_2+z)^2(R_2-z)} - \frac{\nu_2{}^4x^2z}{R_2{}^3(R_2{}^2-z^2)}\right\}$$

$$+ \frac{1}{4\pi}\left\{- \frac{\nu_2{}^3}{R_3(R_3+z)} + \frac{2\nu_3{}^4y^2}{R_3(R_3+z)^2(R_3-z)} - \frac{\nu_3{}^4y^2z}{R_3{}^3(R_3{}^2-z^2)}\right\} \tag{3.9.5}$$

$$\sigma_{yz} = \frac{1}{4\pi}\left\{\frac{2\nu_1{}^4xy}{R_1(R_1+z)^2(R_1-z)} - \frac{\nu_1{}^4xyz}{R_1{}^3(R_1{}^2-z^2)}\right\}$$

$$+ \frac{1}{4\pi}\left\{\frac{2\nu_2{}^4xy}{R_2(R_2+z)^2(R_2-z)} - \frac{\nu_2{}^4xyz}{R_2{}^3(R_2{}^2-z^2)}\right\}$$

$$+ \frac{1}{4\pi}\left\{- \frac{2\nu_3{}^4xy}{R_3(R_3+z)^2(R_3-z)} + \frac{\nu_3{}^4xyz}{R_3{}^3(R_3{}^2-z^2)}\right\} \tag{3.9.6}$$

$$\sigma_{zz} = \frac{1}{4\pi}\{\frac{\nu_1{}^4x}{R_1{}^3}\} + \frac{1}{4\pi}\{\frac{\nu_2{}^4x}{R_2{}^3}\} \tag{3.9.7}$$

$$\sigma_{xx} = \frac{1}{4\pi\nu_3{}^2(k_1+1)}\{-\frac{\nu_1{}^2\nu_3{}^2x(k+1)}{R_1{}^3} + \frac{\nu_1{}^4y}{R_1(R_1{}^2-z^2)} - \frac{2\nu_1{}^4yR_1}{(R_1{}^2-z^2)^2} - \frac{4\nu_1{}^6x^2y}{R_1(R_1{}^2-z^2)^2}$$

$$+ \frac{\nu_1{}^4y}{R_1{}^3} - \frac{\nu_1{}^6x^2y}{R_1{}^3(R_1{}^2-z^2)} + \frac{8\nu_1{}^6x^2yR_1}{(R_1{}^2-z^2)^3}\}$$

$$+ \frac{1}{4\pi\nu_3{}^2(k_1+1)}\{-\frac{\nu_2{}^2\nu_3{}^2x(k_1+1)}{R_2{}^3} + \frac{k_1\nu_2{}^4y}{R_2(R_2{}^2-z^2)} - \frac{2k_1\nu_2{}^4yR_2}{(R_2{}^2-z^2)^2}$$

$$- \frac{4k_1\nu_2{}^6x^2y}{R_2(R_2{}^2-z^2)^2} + \frac{k_1\nu_2{}^4y}{R_2{}^3} - \frac{k_1\nu_2{}^6x^2y}{R_2{}^3(R_2{}^2-z^2)} + \frac{8k_1\nu_2{}^6x^2yR_2}{(R_2{}^2-z^2)^3}\}$$

$$+ \frac{1}{4\pi}\{-\frac{\nu_3{}^2x}{R_3(R_3{}^2-z^2)} + \frac{2\nu_3{}^2xR_3}{(R_3{}^2-z^2)^2} + \frac{4\nu_3{}^4xy^2}{R_3(R_3{}^2-z^2)^2}$$

$$- \frac{8\nu_3{}^4xy^2R^3}{(R_3{}^2-z^2)^3} + \frac{\nu_3{}^4xy^2}{R_3{}^3(R_3{}^2-z^2)}\} \tag{3.9.8}$$

$$\sigma_{yy} = \frac{1}{4\pi\nu_3{}^2(k_1+1)}\{-\frac{\nu_1{}^2\nu_3{}^2x(k_1+1)}{R_1{}^3} + \frac{\nu_1{}^4x}{R_1(R_1{}^2-z^2)} - \frac{2\nu_1{}^4xR_1}{(R_1{}^2-z^2)^2} - \frac{4\nu_1{}^6xy^2}{R_1(R_1{}^2-z^2)^2}$$

$$+ \frac{\nu_1{}^4x}{R_1{}^3} - \frac{\nu_1{}^6xy^2}{R_1{}^3(R_1{}^2-z^2)} + \frac{8\nu_1{}^6xy^2R_1}{(R_1{}^2-z^2)^3}\}$$

$$+ \frac{1}{4\pi\nu_3{}^2(k_1+1)}\{-\frac{\nu_2{}^2\nu_3{}^2x(k_1+1)}{R_2{}^3} + \frac{k_1\nu_2{}^4x}{R_2(R_2{}^2-z^2)} - \frac{2k_1\nu_2{}^4xR_2}{(R_2{}^2-z^2)^2}$$

$$- \frac{4k_1\nu_2{}^6xy^2}{R_2(R_2{}^2-z^2)^2} + \frac{k_1\nu_2{}^4x}{R_2{}^3} - \frac{k_1\nu_2{}^6xy^2}{R_2{}^3(R_2{}^2-z^2)} + \frac{8k_1\nu_2{}^6xy^2R_2}{(R_2{}^2-z^2)^3}\}$$

$$+ \frac{1}{4\pi}\left\{ \frac{\nu_3{}^2 x}{R_3(R_3{}^2-z^2)} - \frac{2\nu_3{}^2 x R_3}{(R_3{}^2-z^2)^2} - \frac{4\nu_3{}^4 x y^2}{R_3(R_3{}^2-z^2)^2} + \frac{8\nu_3{}^4 x y^2 R_3}{(R_3{}^2-z^2)^3} - \frac{\nu_3{}^4 x y^2}{R_3{}^3(R_3{}^2-z^2)} \right\} \tag{3.9.9}$$

$$\sigma_{xy} = \frac{1}{2\pi\nu_3{}^2(k_1+1)}\left\{- \frac{\nu_1{}^6 x^2 y}{R_1(R_1{}^2-z^2)^2} - \frac{\nu_1{}^6 x^2 y}{R_1{}^3(R_1{}^2-z^2)} + \frac{\nu_1{}^4 y}{R_1(R_1{}^2-z^2)}\right.$$

$$\left. + \frac{8\nu_1{}^6 x^2 y R_1}{(R_1{}^2-z^2)^3} - \frac{3\nu_1{}^6 x^2 y}{R_1(R_1{}^2-z^2)^2} - \frac{2\nu_1{}^4 y R_1}{(R_1{}^2-z^2)^2} \right\}$$

$$+ \frac{1}{2\pi\nu_3{}^2(k_1+1)}\left\{- \frac{k_1\nu_2{}^6 x^2 y}{R_2(R_2{}^2-z^2)^2} - \frac{k_1\nu_2{}^6 x^2 y}{R_2{}^3(R_2{}^2-z^2)} + \frac{k_1\nu_2{}^4 y}{R_2(R_2{}^2-z^2)}\right.$$

$$\left. + \frac{8k_1\nu_2{}^6 x^2 y R_2}{(R_2{}^2-z^2)^3} - \frac{3k_1\nu_2{}^6 x^2 y}{R_2(R_2{}^2-z^2)^2} - \frac{2k_1\nu_2{}^4 y R_2}{(R_2{}^2-z^2)^2} \right\}$$

$$+ \frac{1}{4\pi}\left\{- \frac{8\nu_3{}^4 y R_3(x^2-y^2)}{(R_3{}^2-z^2)^3} + \frac{2\nu_3{}^4 x^2 y}{R_3(R_3{}^2-z^2)^2} + \frac{\nu_3{}^4 y(x^2-y^2)}{R_3{}^3(R_3{}^2-z^2)} - \frac{6\nu_3{}^4 y^3}{R_3(R_3{}^2-z^2)^2}\right.$$

$$\left. - \frac{4\nu_3{}^2 y R_3}{(R_3{}^2-z^2)^2} + \frac{4\nu_3{}^2 y}{R_3(R_3{}^2-z^2)} \right\} \tag{3.9.10}$$

3.10 Discussion

The resultant forces in the principal directions for the singularities presented here can easily be found by using the equations derived in Chapter 2.

Moreover, all of these solutions have been found to be valid for the special case of $(b + y) = 0$ where $\nu_1 = \nu_2$ etc. (see Equations (2.3.1)).

It is important to note that the expressions for displacements and stresses throughout this Chapter are written in such a manner that the first term in each expression is related to ϕ_1, the second term to ϕ_2, and the third term, if any, to ψ. This will be important in Chapters 4 and 5 when we deal with the solution of sub-surface problems.

4. SOLUTIONS OF PREVIOUSLY-SOLVED TRANSVERSELY-ISOTROPIC HALF-SPACE PROBLEMS

4.1 Concentrated Vertical Unit-Force on a Half-Space

Usually referred to as the "Boussinesq Problem" (see Figure 4.1), this problem for the case of transverse isotropy has been solved by E. Kröner (1953), Hu (1954), Woo and Shield (1962), and Pan and Chou (1976), by methods other than the singularities approach. The final solutions are all identical to those which are presented here.

The solution, using the method of singularities, can be generated by the realization that it should have some characteristics of a Kelvin force, in that the only singularitity in the lower half-space is at z = 0, the resultant force on any hemisphere in the lower half-space with center at the origin is of a unit magnitude in the z-direction, and the singularity at the origin is $O(\frac{1}{R^2})$. Since the Kelvin force alone does not clear the boundary z=0 of tractions, one must seek further rotationally symmetric singularitites in the upper half-space of order lower than $O(\frac{1}{R^2})$. A line of dilitation has the required symmetry and is $O(\frac{1}{R})$. Superposing these singularities (with the line of dilatation having the proper "magnitude"), clears the boundary z=0. The force resultant was obtained from the equations in Chapter 2 and the "magnitude" of both singularities were adjusted so that the total resultant was equal to a unit-force.

In retrospect it can be seen that the identical solution could be generated by superposing the following:

a. ϕ, portion of a line of dilatation extending from the origin to infinity along the z-axis and having a magnitude of

$$\frac{\nu_1(k - 1)}{(k_1 - \nu_1^2)(\nu_1 - \nu_2)}$$

and,

b. ϕ, portion of the same line of dilitation having a magnitude of

$$-\frac{\nu_2(k_1 - 1)}{(k_1\nu_2^2 - 1)(\nu_1 - \nu_2)}$$

The Kelvin force seems to be omitted but it is realized that at the origin the line of dilatation produces the force resultant required to solve the problem. It is also important to note that separation of the potential functions provides a shorter and equivalent route to the problem's solution. This fact is very helpful in solving more complex problems.

For the Boussinesq problem, the resulting potentials are:

$$\phi_1 = \frac{\nu_1}{2\pi\mu(k_1+1)(\nu_1 - \nu_2)} \{\log (R_1 - z)\}$$

$$\phi_2 = \frac{-k_1\nu_2}{2\pi\mu(k_1+1)(\nu_1 - \nu_2)} \{\log (R_2 - z)\}$$

$$\psi = 0$$

The derived displacements become:

$$U_1 = \frac{1}{2\pi\mu(k_1+1)(\nu_1 - \nu_2)} \left\{ \frac{\nu_1{}^3 x}{R_1(R_1 - z)} - \frac{k_1\nu_2{}^3 x}{R_2(R_2 - z)} \right\}$$

$$U_2 = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\left\{\frac{\nu_1{}^3 y}{R_1(R_1-z)} - \frac{k_1\nu_2{}^3 y}{R_2(R_2-z)}\right\}$$

$$U_3 = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\left\{\frac{k_1\nu_1}{R_1} - \frac{\nu_2}{R_2}\right\} \tag{4.1.2}$$

The stresses are:

$$\sigma_{xz} = \frac{1}{2\pi(\nu_1-\nu_2)}\left\{-\frac{\nu_1{}^3 x}{R_1{}^3} + \frac{\nu_2{}^3 x}{R_2{}^3}\right\} \tag{4.1.3}$$

$$\sigma_{yz} = \frac{1}{2\pi(\nu_1-\nu_2)}\left\{-\frac{\nu_1{}^3 y}{R_1{}^3} + \frac{\nu_2{}^3 y}{R_2{}^3}\right\} \tag{4.1.4}$$

$$\sigma_{zz} = \frac{1}{2\pi(\nu_1-\nu_2)}\left\{-\frac{\nu_1{}^3 z}{R_1{}^3} + \frac{\nu_2{}^3 z}{R_2{}^3}\right\} \tag{4.1.5}$$

$$\sigma_{xx} = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\left\{-\frac{2bk_1\nu_1(\nu_1{}^2-1)}{R_1(R_1+z)} + \frac{bk_1\nu_1 z(\nu_1{}^2-1)}{R_1{}^3} + \frac{2b\nu_2(\nu_2{}^2-1)}{R_2(R_2+z)}\right.$$

$$\left. - \frac{b\nu_2 z(\nu_2{}^2-1)}{R_2{}^3}\right\} + \frac{1}{2\pi(\nu_1-\nu_2)}\left\{-\frac{2\nu_1(\nu_1{}^2-1)}{R_1(R_1+z)}\right.$$

$$\left. + \frac{\nu_1{}^3 z}{R_1{}^3} + \frac{2\nu_2(\nu_2{}^2-1)}{R_2(R_2+z)} - \frac{\nu_2{}^3 z}{R_2{}^3}\right\}$$

$$+ \frac{1}{\pi\nu_3{}^2(k_1+1)(\nu_1-\nu_2)}\left\{-\frac{\nu_1{}^3}{R_1(R_1+z)} + \frac{\nu_1{}^5 x^2}{R_1{}^2(R_1+z)^2} + \frac{\nu_1{}^5 x^2}{R_1{}^3(R_1+z)}\right.$$

$$\left. + \frac{k_1\nu_2{}^3}{R_2(R_2+z)} - \frac{k_1\nu_2{}^5 x^2}{R_2{}^2(R_2+z)^2} - \frac{k_1\nu_2{}^5 x^2}{R_2{}^3(R_2+z)}\right\} \tag{4.1.6}$$

$$\sigma_{yy} = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\{-\frac{2bk_1\nu_1(\nu_1{}^2-1)}{R_1(R_1+z)} + \frac{bk_1\nu_1 z(\nu_1{}^2-1)}{R_1{}^3} + \frac{2b\nu_2(\nu_2{}^2-1)}{R_2(R_2+z)}$$

$$- \frac{b\nu_2 z(\nu_2{}^2-1)}{R_2{}^3}\} + \frac{1}{2\pi(\nu_1-\nu_2)}\{-\frac{2\nu_1(\nu_1{}^2-1)}{R_1(R_1+z)} + \frac{\nu_1{}^3 z}{R_1{}^3} + \frac{2\nu_2(\nu_2{}^2-1)}{R_2(R_2+z)}$$

$$- \frac{\nu_2{}^3 z}{R_2{}^3}\} + \frac{1}{\pi\nu_3{}^2(k_1+1)(\nu_1-\nu_2)}\{-\frac{\nu_1{}^3}{R_1(R_1+z)} + \frac{\nu_1{}^5 y^2}{R_1{}^2(R_1+z)^2}$$

$$+ \frac{\nu_1{}^5 y^2}{R_1{}^3(R_1+z)} + \frac{k_1\nu_2{}^3}{R_2(R_2+z)} - \frac{k_1\nu_2{}^5 y^2}{R_2{}^2(R_2+z)^2} - \frac{k_1\nu_2{}^S y^2}{R_2{}^3(R_2+z)}\} \qquad (4.1.7)$$

$$\sigma_{xy} = \frac{1}{\pi\nu_3{}^2(k_1+1)(\nu_1-\nu_2)}\{-\frac{2\nu_1{}^5\, xy}{R_1(R_1+z)(R_1-z)^2} - \frac{\nu_1{}^5 xyz}{R_1{}^3(R_1{}^2-z^2)}$$

$$+ \frac{2\nu_2{}^S xy}{R_2(R_2+z)(R_2-z)^2} + \frac{\nu_2{}^S xyz}{R_2{}^3(R_2{}^2-z^2)}\} \qquad (4.1.8)$$

4.2 Concentrated Tangential Unit-Force on a Half-Space

This problem (see Figure 4.2) usually referred to as the "Cerruti Problem", was also solved by other methods for the case of a transversely isotropic halfspace by E. Kroner (1953), Hu (1954), Woo and Shield (1962), and, Pan and Chow (1976).

Using the mehtod of singularities, the solution can be generated by realizing that it should contain a Kelvin force in the x-direction, $O(\frac{1}{R^2})$, which is itself odd in x and even in y. Therefore, a singularity of lower order which has this same symmetry, and which acts only in the upper half-space, must be sought. It was found that a line of rotation along the z-axis whose centers of rotation produce moments about the y-axis, has these characteristics.

It was found that superposing the following yielded the desired results:

a. Unit Kelvin force at the origin, acting in the positive x-direction

b. ϕ_1 and ϕ_2 portions of a line of rotation (whose resulting moments are about the positive y-axis) from the origin to infinity and having a magnitude of

$$\frac{\nu_1 - k_1\nu_2}{(k_1-1)(\nu_1+\nu_2)}$$

c. ψ portion of the same line of rotation and having a magnitude of

$$\frac{-2(\nu_1 - k_1\nu_2)}{(k_1-1)(\nu_1+\nu_2)}$$

d. The force resultant in the x-direction was calculated and adjusted for a unit force.

In retrospect it was also found that identical results can be obtained by superposing:

a. ϕ_1 portion of a line of rotation (whose resulting moments are about the positive y-axis) extending from the origin to infinity along the z-axis having a magnitude of $-\dfrac{2\nu_2}{(\nu_1 - \nu_2)}$,

b. ϕ_2 portion of the same line of rotation but having a magnitude of $\dfrac{2\nu_1}{(\nu_1 - \nu_2)}$.

c. ψ portion of the same line of rotation but having a magnitude of 2.

The resulting potentials are:

$$\phi_1 = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\{ \frac{\nu_1{}^2\nu_2 x}{R_1-z} \}$$

$$\phi_2 = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\{- \frac{k_1\nu_1\nu_2{}^2 x}{R_2-z} \}$$

$$\psi = \frac{1}{2\pi\mu}\{- \frac{\nu_3{}^2 y}{R_3-z} \} \qquad (4.2.1)$$

The displacements are:

$$U_1 = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\{- \frac{\nu_1{}^4\nu_2 x^2}{R_1(R_1-z)^2} + \frac{\nu_1{}^2\nu_2}{(R_1-z)} + \frac{k_1\nu_1\nu_2{}^4 x^2}{R_2(R_2-z)^2}$$

$$- \frac{k_1\nu_1\nu_2{}^2}{(R_2-z)} \} + \frac{1}{2\pi\mu}\{ \frac{\nu_3{}^4 y^2}{R_3(R_3-z)^2} - \frac{\nu_3{}^2}{(R_3-z)} \}$$

$$U_2 = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\{- \frac{\nu_1{}^4\nu_2 xy}{R_1(R_1-z)^2} + \frac{k_1\nu_1\nu_2{}^4 xy}{R_2(R_2-z)^2} \}$$

$$+ \frac{1}{2\pi\mu}\{- \frac{\nu_3{}^4 xy}{R_3(R_3-z)^2} \}$$

$$U_3 = \frac{1}{2\pi\mu(k_1+1)(\nu_1-\nu_2)}\{- \frac{k_1\nu_1{}^2\nu_2 x}{R_1(R_1-z)} + \frac{\nu_1\nu_2{}^2 x}{R_2(R_2-z)} \} \qquad (4.2.2)$$

The stresses are:

$$\sigma_{xz} = \frac{-1}{2\pi(\nu_1-\nu_2)}\{ \frac{\nu_1{}^4\nu_2 x^2}{R_1{}^2(R_1-z)^2} - \frac{\nu_1{}^2\nu_2}{R_1(R_1-z)} + \frac{\nu_1{}^4\nu_2 x^2}{R_1{}^3(R_1-z)}$$

$$- \frac{\nu_1\nu_2{}^4x^2}{R_2{}^2(R_2-z)^2} + \frac{\nu_1\nu_2{}^2}{R_2(R_2-z)} - \frac{\nu_1\nu_2{}^4x^2}{R_2{}^3(R_2-z)}\}$$

$$+ \frac{1}{2\pi}\{- \frac{\nu_3{}^4y^2}{R_3{}^2(R_3-z)^2} + \frac{\nu_3{}^2}{R_3(R_3-z)} - \frac{\nu_3{}^4y^2}{R_3{}^3(R_3-z)}\} \qquad (4.2.3)$$

$$\sigma_{yz} = \frac{1}{2\pi(\nu_1-\nu_2)}\{ \frac{\nu_1{}^4\nu_2xy}{R_1{}^2(R_1-z)^2} + \frac{\nu_1{}^4\nu_2xy}{R_1{}^3(R_1-z)} - \frac{\nu_1\nu_2{}^4xy}{R_2{}^2(R_2-z)^2} - \frac{\nu_1\nu_2{}^4xy}{R_2{}^3(R_2-z)}\}$$

$$+ \frac{1}{2\pi}\{ \frac{\nu_3{}^4xy}{R_3{}^2(R_3-z)^2} + \frac{\nu_3{}^4xy}{R_3{}^3(R_3-z)}\} \qquad (4.2.4)$$

$$\sigma_{zz} = \frac{1}{2\pi(\nu_1-\nu_2)}\{ \frac{\nu_1{}^4\nu_2x}{R_1{}^3} - \frac{\nu_1\nu_2{}^4x}{R_2{}^3}\} \qquad (4.2.5)$$

$$\sigma_{xx} = \frac{1}{2\pi(\nu_1-\nu_2)}\{- \frac{\nu_1{}^2\nu_2x}{R_1{}^3} + \frac{\nu_1\nu_2{}^2x}{R_2{}^3}\}$$

$$+ \frac{1}{\pi}\{- \frac{2\nu_3{}^4xy^2}{R_3{}^2(R_3-z)^3} - \frac{\nu_3{}^4xy^2}{R_3{}^3(R_3-z)^2} + \frac{\nu_3{}^2x}{R_3(R_3-z)^2}\}$$

$$+ \frac{1}{\pi\nu_3{}^2(k_1+1)(\nu_1-\nu_2)}\{- \frac{2\nu_1{}^6\nu_2xy^2}{R_1{}^2(R_1-z)^3} - \frac{\nu_1{}^6\nu_2xy^2}{R_1{}^3(R_1-z)^2} + \frac{\nu_1{}^4\nu_2x}{R_1(R_1-z)^2}$$

$$+ \frac{2k_1\nu_1\nu_2{}^6xy^2}{R_2{}^2(R_2-z)^3} + \frac{k_1\nu_1\nu_2{}^6xy^2}{R_2{}^3(R_2-z)^2} - \frac{k_1\nu_1\nu_2{}^4x}{R_2(R_2-z)^2}\} \qquad (4.2.6)$$

$$\sigma_{yy} = \frac{1}{2\pi(\nu_1-\nu_2)}\{- \frac{\nu_1{}^2\nu_2x}{R_1{}^3} + \frac{\nu_1\nu_2{}^2x}{R_2{}^3}$$

$$+ \frac{1}{\pi} \{ \frac{2\nu_3{}^4 xy^2}{R_3{}^2(R_3-z)^3} + \frac{\nu_3{}^4 xy^2}{R_3{}^3(R_3-z)^2} - \frac{\nu_3{}^2 x}{R_3(R_3-z)^2} \}$$

$$+ \frac{1}{\pi\nu_3{}^2(k_1+1)(\nu_1-\nu_2)} \{ - \frac{2\nu_1{}^6\nu_2 x^3}{R_1{}^2(R_1-z)^3} - \frac{\nu_1{}^6\nu_2 x^3}{R_1{}^3(R_1-z)^2} + \frac{3\nu_1{}^4\nu_2 x}{R_1(R_1-z)^2}$$

$$+ \frac{2k_1\nu_1\nu_2{}^6 x^3}{R_2{}^2(R_2-z)^3} + \frac{k_1\nu_1\nu_2{}^6 x^3}{R_2{}^3(R_2-z)^2} - \frac{3k_1\nu_1\nu_2{}^4 x}{R_2(R_2-z)^2} \} \qquad (4.2.7)$$

$$\sigma_{xy} = \frac{1}{\pi\nu_3{}^2(k_1+1)(\nu_1-\nu_2)} \{ - \frac{\nu_1{}^4\nu^2 y}{R_1(R_1-z)^2} + \frac{2\nu_1{}^6\nu_2 x^2 y}{R_1{}^2(R_1-z)^3} + \frac{\nu_1{}^6\nu_2 x^2 y}{R_1{}^3(R_1-z)^2}$$

$$+ \frac{k_1\nu_1\nu_2{}^4 y}{R_2(R_2-z)^2} - \frac{2k_1\nu_1\nu_2{}^6 x^2 y}{R_2{}^2(R_2-z)^3} - \frac{k_1\nu_1\nu_2{}^6 x^2 y}{R_2{}^3(R_2-z)^2} \}$$

$$+ \frac{1}{2\pi} \{ \frac{2\nu_3{}^4 y(x^2-y^2)}{R_3{}^2(R_3-z)^3} + \frac{2\nu_3{}^2 y}{R_3(R_3-z)^2} + \frac{\nu_3{}^4 y(x^2-y^2)}{R_3{}^3(R_3-z)^2} \} \qquad (4.2.8)$$

4.3 Concentrated Vertical Unit-Force Beneath the Surface of a Half-Space

The solution of this problem (see Figure 4.3), referred to as the "Mindlin Problem" in the literature, has been solved for transverse isotropy by Shield (1951) by another approach.

For guidance in the construction of this solution by the singularities approach, it is observed that the only permissible singluarity in the lower half-space is the Kelvin force. Analogy with simpler elliptic fields and the desire to clear the boundary of normal stresses lead to the placement of a Kelvin force singularity of opposite sign at the "image point" outside the boundary. Finally, the realization that the limit solution must approach the Boussinesq solution as the point force approaches the boundary, leads one to utilize the potentials of a line of dilatation to clear the boundary of the remaining tractions.

A solution was obtained by superposition (letting the force act at a point "S" beneath the surface) of the following:

a. Unit force in the negative z-direction along the z-axis and acting at a point a distance "S" below the surface of the half-space.

b. Unit force in the positive z-direction along the z-axis and acting at a point a distance "S" above the surface of the half-space, that is, at the "image point".

c. ϕ_1, portion of a line of dilitation extending from "S" to infinity along the positive z-axis and having a magnitude of $\dfrac{-k_1\nu_1}{(k_1 - \nu_1^2)(\nu_1 - \nu_2)}$,

d. ϕ_2 portion of the same line of dilitation and having a magnitude of $\dfrac{-\nu_2}{(k_1\nu_2^2-1)(\nu_1-\nu_2)}$

e. ϕ_1 portion of a line of dilitation extending from $\dfrac{S\nu_1}{\nu_2}$ to infinity along the positive.

z-axis and having a magnitude of $\dfrac{\nu_1}{(k_1-\nu_1^2)(\nu_1-\nu_2)}$

f. ϕ_2 portion of a line of dilitation extending from $\dfrac{S\nu_2}{\nu_1}$ to infinity along the positive z-axis and having a magnitude of $\dfrac{k_1\nu_2}{(k_1\nu_2^2-1)(\nu_1-\nu_2)}$

Although the force resultant can be calculated by the method and formulas outlined in Chapter 2, it clearly retains the unit magnitude of the Kelvin force.

Letting:

$$M_1{}^* = [\nu_1{}^2 r^2 + (z - \frac{s\nu_1}{\nu_2})^2]^{\frac{1}{2}}$$

$$m_1 = M_1{}^* + z - \frac{s\nu_1}{\nu_2}$$

$$N_2{}^* = [\nu_2{}^2 r^2 + (z - \frac{s\nu_2}{\nu_1})^2]^{\frac{1}{2}}$$

$$n_2 = N_2{}^* + z - \frac{s\nu_2}{\nu_1} \qquad (4.3.2)$$

$$Q_\ell{}^* = [\nu_\ell{}^2 r^2 + (z-s)^2]^{\frac{1}{2}} \quad (\ell = 1,2,3)$$

$$g_\ell = Q_\ell{}^* + z-s \quad (\ell = 1,2,3) \qquad (4.3.3)$$

$$R_\ell{}^* = [\nu_\ell{}^2 r^2 + (z + s)^2]^{\frac{1}{2}} \quad (\ell = 1,2,3)$$

$$r_\ell = R_\ell{}^* + z + s \quad (\ell = 1,2,3) \qquad (4.3.4)$$

the derived potential functions are:

$$\phi_1 = \frac{1}{8\pi\mu(k_1{}^2-1)}\{-k_1 \log \left(\frac{R_1{}^* +z+s}{R_1{}^* -z-s}\right) + k_1 \log \left(\frac{Q_1{}^* +z-s}{Q_1{}^* -z+s}\right)$$

$$- \frac{4k_1\nu_1}{(\nu_1-\nu_2)} \log (Q_1^* +z-s) + \frac{4\nu_1}{(\nu_1-\nu_2)} \log (M_1^* +z - \frac{s\nu_1}{\nu_2})\} \quad (4.3.5)$$

$$\phi_2 = \frac{1}{8\pi\mu(k_1{}^2-1)}\{ k_1 \log (\frac{R_2^* +z+s}{R_2^* -z-s}) - k_1 \log (\frac{Q_2^* +z-s}{Q_2^* -z+s})$$

$$- \frac{4k_1\nu_2}{(\nu_1-\nu_2)} \log (Q_2^* +z-s) + \frac{4k_1{}^2\nu_2}{(\nu_1-\nu_2)} \log (N_2^* +z - \frac{s\nu_2}{\nu_1})\} \quad (4.3.6)$$

$$\psi = 0 \quad (4.3.7)$$

The displacements become:

$$U_1 = \frac{1}{4\pi\mu(k_1{}^2-1)}\{- \frac{k_1\nu_1{}^2x(z-s)}{Q_1^*q_1(Q_1^* -z+s)} + \frac{k_1\nu_2{}^2x(z-s)}{Q_2^*q_2(Q_2^* -z+s)}$$

$$+ \frac{k_1\nu_1{}^2x(z+s)}{R_1^*r_1(R_1^*-z-s)} - \frac{k_1\nu_2{}^2x(z+s)}{R_2^*r_2(R_2^*-z-s)}\}$$

$$+ \frac{1}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)}\{- \frac{k_1\nu_1{}^3x}{Q_1^*q_1} - \frac{k_1\nu_2{}^3x}{Q_2^*q_2} + \frac{\nu_1{}^3x}{M_1^*m_1} + \frac{k_1{}^2\nu_2{}^3x}{N_2^*n_2}\} \quad (4.3.8)$$

$$U_2 = \frac{1}{4\pi\mu(k_1{}^2-1)}\{- \frac{k_1\nu_1{}^2y(z-s)}{Q_1^*q_1(Q_1^*-z+s)} + \frac{k_1\nu_2{}^2y(z-s)}{Q_2^*q_2(Q_2^*-z+s)}$$

$$+ \frac{k_1\nu_1{}^2y(z+s)}{R_1^*r_1(R_1^*-z-s)} - \frac{k_1\nu_2{}^2y(z+s)}{R_2^*r_2(R_2^*-z-s)}\}$$

$$+\frac{1}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)}\{-\frac{k_1\nu_1{}^3y}{Q_1{}^*q_1}-\frac{k_1\nu_2{}^3y}{Q_2{}^*q_2}$$

$$+\frac{\nu_1{}^3y}{M_1{}^*m_1}+\frac{k_1{}^2\nu_2{}^3y}{N_2{}^*n_2}\} \tag{4.3.9}$$

$$U_3=\frac{1}{4\pi\mu(k_1{}^2-1)}\{\frac{k_1{}^2}{Q_1{}^*}-\frac{1}{Q_2{}^*}-\frac{k_1{}^2}{R_1{}^*}+\frac{1}{R_2{}^*}\}$$

$$+\frac{1}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)}\{-\frac{k_1{}^2\nu_1}{Q_1{}^*}-\frac{\nu_2}{Q_2{}^*}+\frac{k_1\nu_1}{M_1{}^*}+\frac{k_1\nu_2}{N_2{}^*}\} \tag{4.3.10}$$

and the stresses are:

$$\sigma_{xz}=\frac{1}{4\pi(k_1-1)}\{-\frac{k_1\nu_1{}^2x}{Q_1{}^{*3}}+\frac{\nu_2{}^2x}{Q_2{}^{*3}}+\frac{k_1\nu_1{}^2x}{R_1{}^{*3}}-\frac{\nu_2{}^2x}{R_2{}^{*3}}\}$$

$$+\frac{1}{2\pi(k_1-1)(\nu_1-\nu_2)}\{\frac{k_1\nu_1{}^3x}{Q_1{}^{*3}}+\frac{\nu_2{}^3x}{Q_2{}^{*3}}-\frac{\nu_1{}^3x}{M_1{}^{*3}}-\frac{k_1\nu_2{}^3\ x}{N_2{}^{*3}}\} \tag{4.3.11}$$

$$\sigma_{yz}=\frac{1}{4\pi(k_1-1)}\{-\frac{k_1\nu_1{}^2y}{Q_1{}^{*3}}+\frac{\nu_2{}^2\ y}{Q_2{}^{*3}}+\frac{k_1\nu_1{}^2y}{R_1{}^{*3}}-\frac{\nu_2{}^2y}{R_2{}^{*3}}\}$$

$$+\frac{1}{2\pi(k_1-1)(\nu_1-\nu_2)}\{\frac{k_1\nu_1{}^3y}{Q_1{}^{*3}}+\frac{\nu_2{}^3y}{Q_2{}^{*3}}-\frac{\nu_1{}^3y}{M_1{}^{*3}}-\frac{k_1\nu_2{}^3y}{N_2{}^{*3}}\} \tag{4.3.12}$$

$$\sigma_{zz}=\frac{1}{4\pi(k_1-1)}\{-\frac{k_1\nu_1{}^2(z-s)}{Q_1{}^{*3}}+\frac{\nu_2{}^2(z-s)}{Q_2{}^{*3}}+\frac{k_1\nu_1{}^2(z+s)}{R_1{}^{*3}}-\frac{\nu_2{}^2(z+s)}{R_2{}^{*3}}\}$$

$$+\frac{1}{2\pi\mu(k_1-1)(\nu_1-\nu_2)}\{\frac{k_1\nu_1{}^3(z-s)}{Q_1{}^{*3}}+\frac{\nu_2{}^3(z-s)}{Q_2{}^{*3}}-\frac{\nu_1{}^3(z-\frac{s\nu_1}{\nu_2})}{M_1{}^{*3}}$$

$$- \frac{k_1 \nu_2{}^3 (z - \frac{s\nu_2}{\nu_2})}{N_2{}^{*3}} \tag{4.3.13}$$

$$\sigma_{xx} = \frac{1}{4\pi(k_1-1)} \{ \frac{k_1(z-s)}{Q_1{}^{*3}} - \frac{(z-s)}{Q_2{}^{*3}} - \frac{k_1(z+s)}{R_1{}^{*3}} + \frac{(z+s)}{R_2{}^{*3}} \}$$

$$+ \frac{a}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)} \{ - \frac{k_1\nu_1{}^3}{Q_1{}^* q_1} - \frac{k_1\nu_1{}^3(z-s)^2}{Q_1{}^{*3} q_1} + \frac{k_1\nu_1{}^5 r^2}{Q_1{}^{*2} q_1{}^2} - \frac{k_1\nu_2{}^3}{Q_2{}^* q_2}$$

$$- \frac{k_1\nu_2{}^3(z-s)^2}{Q_2{}^{*3} q_2} + \frac{k_1\nu_2{}^5 r^2}{Q_2{}^{*2} q_2{}^2} + \frac{\nu_1{}^3}{M_1{}^* m_1} + \frac{\nu_1{}^3(z - \frac{s\nu_1}{\nu_2})^2}{M_1{}^{*3} m_1} - \frac{\nu_1{}^5 r^2}{M_1{}^{*2} m_1}$$

$$+ \frac{k_1{}^2\nu_2{}^3}{N_2{}^* n_2} + \frac{k_1{}^2\nu_2{}^3(z - \frac{s\nu_2}{\nu_1})^2}{N_2{}^{*3} n_2} - \frac{k_1{}^2\nu_2{}^5 r^2}{N_2{}^{*2} n_2{}^2} \}$$

$$+ \frac{b}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)} \{ \frac{k_1{}^2\nu_1(z-s)}{Q_1{}^{*3}} + \frac{\nu_2(z-s)}{Q_2{}^{*3}} - \frac{k_1\nu_1(z - \frac{s\nu_1}{\nu_2})}{M_1{}^{*3}}$$

$$- \frac{k_1\nu_2(z - \frac{s\nu_2}{\nu_1})}{N_2{}^{*3}} \}$$

$$+ \frac{1}{2\pi\nu_3{}^2(k_1{}^2-1)} \{ \frac{k_1\nu_1{}^2(z-s)}{Q_1{}^*[Q_1{}^{*2}-(z-s)^2]} - \frac{2k_1\nu_1{}^4 y^2(z-s)}{Q_1{}^*[Q_1{}^{*2}-(z-s)^2]^2}$$

$$- \frac{k_1\nu_1{}^4 y^2(z-s)}{Q_1{}^{*3}[Q_1{}^{*2}-(z-s)^2]} - \frac{k_1\nu_2{}^2(z-s)}{Q_2{}^*[Q_2{}^{*2}-(z-s)^2]}$$

$$+ \frac{2k_1\nu_2{}^4y^2(z-s)}{Q_2{}^*[Q_2{}^{*2}-(z-s)^2]^2} + \frac{k_1\nu_2{}^4y^2(z-s)}{Q_2{}^{*3}[Q_2{}^{*2}-(z-s)^2]}$$

$$- \frac{k_1\nu_1{}^2(z+s)}{R_1{}^*[R_1{}^{*2}-(z+s)^2]} + \frac{2k_1\nu_1{}^4y^2(z+s)}{R_1{}^*[R_1{}^{*2}-(z+s)^2]^2} + \frac{k_1\nu_1{}^4y^2(z+s)}{R_1{}^{*3}[R_1{}^{*2}-(z+s)^2]}$$

$$+ \frac{k_1\nu_2{}^2(z+s)}{R_2{}^*[R_2{}^{*2}-(z+s)^2]} - \frac{2k_1\nu_2{}^4y^2(z+s)}{R_2{}^*[R_2{}^{*2}-(z+s)^2]^2} - \frac{k_1\nu_2{}^4y^2(z+s)}{R_2{}^{*3}[R_2{}^{*2}-(z+s)^2]} \}$$

$$+ \frac{1}{\pi\nu_3{}^2(k_1{}^2-1)(\nu_1-\nu_2)} \{ \frac{k_1\nu_1{}^3}{Q_1{}^*q_1} - \frac{k_1\nu_1{}^5y^2}{Q_1{}^{*3}q_1} - \frac{k_1\nu_1{}^5y^2}{Q_1{}^{*2}q_1{}^2}$$

$$+ \frac{k_1\nu_2{}^3}{Q_2{}^*q_2} - \frac{k_1\nu_2{}^5y^2}{Q_2{}^{*3}q_2} - \frac{k_1\nu_2{}^5y^2}{Q_2{}^{*2}q_2{}^2} - \frac{\nu_1{}^3}{M_1{}^*m_1} + \frac{\nu_1{}^5y^2}{M_1{}^{*3}m_1}$$

$$+ \frac{\nu_1{}^5y^2}{M_1{}^{*2}m_1{}^2} - \frac{k_1{}^2\nu_2{}^3}{N_2{}^*n_2} + \frac{k_1{}^2\nu_2{}^5y^2}{N_2{}^{*3}n_2} + \frac{k_1{}^2\nu_2{}^5y^2}{N_2{}^{*2}n_2{}^2} \} \qquad (4.3.14)$$

$$\sigma_{yy} = \frac{1}{4\pi(k_1-1)} \{ \frac{k_1(z-s)}{Q_1{}^{*3}} - \frac{(z-s)}{Q_2{}^{*3}} - \frac{k_1(z+s)}{R_1{}^{*3}} + \frac{(z+s)}{R_2{}^{*3}} \}$$

$$+ \frac{a}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)} \{ - \frac{k_1\nu_1{}^3}{Q_1{}^*q_1} - \frac{k_1\nu_1{}^3(z-s)^2}{Q_1{}^{*3}q_1} + \frac{k_1\nu_1{}^5r^2}{Q_1{}^{*2}q_1{}^2} - \frac{k_1\nu_2{}^3}{Q_2{}^*q_2}$$

$$- \frac{k_1\nu_2{}^3(z-s)^2}{Q_2{}^{*3}q_2} + \frac{k_1\nu_2{}^5r^2}{Q_2{}^{*2}q_2{}^2} + \frac{\nu_1{}^3}{M_1{}^*m_1} + \frac{\nu_1{}^3(z-\frac{s\nu_1}{\nu_2})^2}{M_1{}^{*3}m_1} - \frac{\nu_1{}^5r^2}{M_1{}^{*2}m_1}$$

$$+ \frac{k_1{}^2\nu_2{}^3}{N_2{}^*n_2} + \frac{k_1{}^2\nu_2{}^3(z-\frac{s\nu_2}{\nu_1})^2}{N_2{}^{*3}n_2} - \frac{k_1{}^2\nu_2{}^5r^2}{N_2{}^{*2}n_2{}^2} \}$$

$$+ \frac{b}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)} \{ \frac{k_1{}^2\nu_1(z-s)}{Q_1{}^{*3}} + \frac{\nu_2(z-s)}{Q_2{}^{*3}} - \frac{k_1\nu_1(z-\frac{s\nu_1}{\nu_2})}{M_1{}^{*3}}$$

$$- \frac{k_1\nu_2(z-\frac{s\nu_2}{\nu_1})}{N_2{}^{*3}} \}$$

$$+ \frac{1}{2\pi\nu_3{}^2(k_1{}^2-1)} \{ \frac{k_1\nu_1{}^2(z-s)}{Q_1{}^*[Q_1{}^{*2}-(z-s)^2]} - \frac{2k_1\nu_1{}^4y^2(z-s)}{Q_1{}^*[Q_1{}^{*2}-(z-s)^2]^2}$$

$$- \frac{k_1\nu_1{}^4y^2(z-s)}{Q{}^{*3}[Q{}^{*2}-(z-s)^2]} - \frac{k_1\nu_2{}^2(z-s)}{Q_2{}^*[Q_2{}^{*2}-(z-s)^2]} + \frac{2k_1\nu_2{}^4y^2(z-s)}{Q_2{}^*[Q_2{}^{*2}-(z-s)^2]^2}$$

$$+ \frac{k_1\nu_2{}^4y^2(z-s)}{Q_2{}^{*3}[Q_2{}^{*2}-(z-s)^2]}$$

$$- \frac{k_1\nu_1{}^2(z+s)}{R_1{}^*[R_1{}^{*2}-(z+s)^2]} + \frac{2k_1\nu_1{}^4y^2(z+s)}{R_1{}^*[R_1{}^{*2}-(z+s)^2]^2} + \frac{k_1\nu_1{}^4y^2(z+s)}{R_1{}^{*3}[R_1{}^{*2}-(z+s)^2]}$$

$$+ \frac{k_1\nu_2{}^2(z+s)}{R_2{}^*[R_2{}^{*2}-(z+s)^2]} - \frac{2k_1\nu_2{}^4y^2(z+s)}{R_2{}^*[R_2{}^{*2}-(z+s)^2]^2} - \frac{k_1\nu_2{}^4y^2(z+s)}{R_2{}^{*3}[R_2{}^{*2}-(z+s)^2]} \}$$

$$+ \frac{1}{\pi\nu_3{}^2(k_1{}^2-1)(\nu_1-\nu_2)} \{ \frac{k_1\nu_1{}^3}{Q_1{}^*q_1} - \frac{k_1\nu_1{}^5x^2}{Q_1{}^{*3}q_1} - \frac{k_1\nu_1{}^5x^2}{Q_1{}^{*2}q_1{}^2} + \frac{k_1\nu_2{}^3}{Q_2{}^*q_2}$$

$$- \frac{k_1\nu_2{}^5x^2}{Q_2{}^{*3}q_2} - \frac{k_1\nu_2{}^5x^2}{Q_2{}^{*2}q_2{}^2} - \frac{\nu_1{}^3}{M_1{}^*m_1} + \frac{\nu_1{}^5x^2}{M_1{}^{*3}m_1} + \frac{\nu_1{}^5x^2}{M_1{}^{*2}m_1{}^2} + \frac{k_1{}^2\nu_2{}^3}{N_2{}^*n_2}$$

$$+ \frac{k_1{}^2\nu_2{}^5x^2}{N_2{}^{*3}n_2} + \frac{k_1{}^2\nu_2{}^5x^2}{N_2{}^{*2}n_2{}^2} \} \qquad (4.3.15)$$

$$\sigma_{xy} = \frac{1}{\pi\nu_3^2(k_1^2-1)}\left\{ \frac{2k_1\nu_1^4xy(z-s)}{Q_1^*[Q_1^{*2}-(z-s)^2]^2} + \frac{k_1\nu_1^4xy(z-s)}{Q_1^{*2}[Q_1^{*2}-(z-s)^2]} \right.$$

$$\left. - \frac{2k_1\nu_2^4xy(z-s)}{Q_2^*[Q_2^{*2}-(z-s)^2]^2} - \frac{k_1\nu_2^4xy(z-s)}{Q_2^{*2}[Q_2^{*2}-(z-s)^2]} \right\}$$

$$+ \frac{1}{\pi\nu_3^2(k_1^2-1)(\nu_1-\nu_2)}\left\{ \frac{k_1\nu_1^5xy}{Q_1^{*2}q_1^2} + \frac{k_1\nu_1^5xy}{Q_1^{*3}q_1} + \frac{k_1\nu_2^5xy}{Q_2^{*2}q_2^2} + \frac{k_1\nu_2^5xy}{Q_2^{*3}q_2} \right.$$

$$\left. - \frac{\nu_1^5xy}{M_1^{*2}m_1^2} - \frac{\nu_1^5xy}{M_1^{*3}m_1} - \frac{k_1^2\nu_2^5xy}{N_2^{*2}n_2^2} - \frac{k_1^2\nu_2^5xy}{N_2^{*3}n_2} \right\} \qquad (4.3.16.)$$

5. TANGENTIAL UNIT-FORCE BENEATH THE SURFACE OF A TRANSVERSELY ISOTROPIC HALF-SPACE

5.1 Explanation and Solution

The usefulness of the singularities approach was demonstrated in the solutions presented in Chapter 4. In order to solve the problem of a subsurface horizontal unit force in a transversely isotropic medium, it must be observed that the only permissible singularity in the lower half-space is the Kelvin force. All other singularities must lie in the upper half-space. The realization that the limit solution, as the force approaches the surface, must approach the Cerruti solution, leads one to the use of the potentials of a line of rotation to clear the boundary of normal stresses.

Letting the force act at a point a distance "s" beneath the surface, a solution was effected as follows (see Figure 5.1):

a. Place a unit-force acting in the x-direction at a point a distance "s" (along the z-axis) below the surface.

b. Place a unit force acting in the negative x-direction at a point a distance "s", along the z-axis, above the surface of the half-space. Note that this force acts at the "image point."

c. Superpose a line of rotation extending along the positive z-axis from "s" to infinity, where the ϕ_1 portion of the line has a magnitude of $\frac{2\nu_2}{(k_1-1)(\nu_1-\nu_2)}$, the ϕ_2 portion a magnitude of $\frac{2k_1\nu_1}{(k_1-1)(\nu_1-\nu_2)}$ and the ψ portion a magnitude of 2.

d. Further superpose the ϕ_1 portion of a line of rotation extending along the positive z-axis from a point, $\frac{s\nu_1}{\nu_2}$ to infinity and having a magnitude of $\frac{-2k_1\nu_2}{(k_1-1)(\nu_1-\nu_2)}$.

e. Finally, add the ϕ_2 portion of a line of rotation extending along the positive z-axis from a point $+\frac{s\nu_2}{\nu_1}$ to infinity and having a magnitude of $-\frac{-2\nu_1}{(k_1-1)(\nu_1-\nu_2)}$.

This selection of potentials results in the proper stress distributions which clear the boundary of the half-space and produce the desired resultant unit force in the x-direction applied directly below and at a point "s" below the surface of the half-space.

Adopting the notation introduced in Section 4.3, the potential functions, derived as outlined above are:

$$\phi_1 = \frac{1}{4\pi\mu(k_1^2-1)}\left\{\frac{\nu_1^2 x}{q_1} - \frac{\nu_1^2 x}{r_1}\right\} + \frac{1}{2\pi\mu(k_1^2-1)(\nu_1-\nu_2)}\left\{\frac{\nu_1^2\nu_2 x}{q_1} - \frac{k_1\nu_1^2\nu_2 x}{m_1}\right\}$$

$$\phi_2 = \frac{1}{4\pi\mu(k_1^2-1)}\left\{\frac{k_1^2\nu_2^2 x}{r_2} - \frac{k_1^2\nu_2^2 x}{q_2}\right\} + \frac{1}{2\pi\mu(k_1^2-1)(\nu_1-\nu_2)}\left\{\frac{k_1^2\nu_1\nu_2^2 x}{q_2} - \frac{k_1\nu_1\nu_2^2 x}{n_2}\right\}$$

$$\psi = \frac{1}{4\pi\mu}\left\{\frac{\nu_3^2 y}{r_3} + \frac{\nu_3^2 y}{q_3}\right\} \qquad (5.1.1)$$

The resulting displacements and stresses are:

$$U_1 = \frac{1}{4\pi\mu(k_1^2-1)}\left\{\frac{\nu_1^2}{q_1} - \frac{\nu_1^4 x^2}{Q_1^* q_1^2} - \frac{k_1^2\nu_2^2}{q_2} + \frac{k_1^2\nu_2^4 x^2}{Q_2^* q_2^2} - \frac{\nu_1^2}{r_1}\right.$$

$$+ \frac{\nu_1{}^4 x^2}{R_1^* r_1{}^2} + \frac{k_1{}^2 \nu_2{}^2}{r_2} - \frac{k_1{}^2 \nu_2{}^4 x^2}{R_2^* r_2{}^2} \}$$

$$+ \frac{1}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)} \{ \frac{\nu_1{}^2 \nu_2}{q_1} - \frac{\nu_1{}^4 \nu_2 x^2}{Q_1^* q_1{}^2} + \frac{k_1{}^2 \nu_1 \nu_2{}^2}{q_2} - \frac{k_1{}^2 \nu_1 \nu_2{}^4 x^2}{Q_2^* q_2{}^2}$$

$$- \frac{k_1 \nu_1{}^2 \nu_2}{m_1} + \frac{k_1 \nu_1{}^4 \nu_2 x^2}{M_1^* m_1{}^2} - \frac{k_1 \nu_1 \nu_2{}^2}{n_2} + \frac{k_1 \nu_1 \nu_2{}^4 x^2}{N_2^* n_2{}^2} \}$$

$$+ \frac{1}{4\pi\mu} \{ \frac{\nu_3{}^2}{q_3} - \frac{\nu_3{}^4 y^2}{Q_3^* q_3{}^2} + \frac{\nu_3{}^2}{r_3} - \frac{\nu_3{}^4 y^2}{R_3^* r_3{}^2} \} \qquad (5.1.2)$$

$$U_2 = \frac{1}{4\pi\mu(k_1{}^2-1)} \{ - \frac{\nu_1{}^4 xy}{Q_1^* q_1{}^2} + \frac{k_1{}^2 \nu_2{}^4 xy}{Q_2^* q_2{}^2} + \frac{\nu_1{}^4 xy}{R_1^* r_1{}^2} - \frac{k_1{}^2 \nu_2{}^4 xy}{R_2^* r_2{}^2} \}$$

$$+ \frac{1}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)} \{ - \frac{\nu_1{}^4 \nu_2 xy}{Q_1^* q_1{}^2} - \frac{k_1{}^2 \nu_1 \nu_2{}^4 xy}{Q_2^* q_2{}^2} + \frac{k_1 \nu_1{}^4 \nu_2 xy}{M_1^* m_1{}^2}$$

$$+ \frac{k_1 \nu_1 \nu_2{}^4 xy}{N_2^* n_2{}^2} \} + \frac{1}{4\pi\mu} \{ \frac{\nu_3{}^4 xy}{Q_3^* q_3{}^2} + \frac{\nu_3{}^4 xy}{R_3^* r_3{}^2} \} \qquad (5.1.3)$$

$$U_3 = \frac{1}{4\pi\mu(k_1{}^2-1)} \{ - \frac{k_1 \nu_1{}^2 x}{Q_1^* q_1} + \frac{k_1 \nu_2{}^2 x}{Q_2^* q_2} + \frac{k_1 \nu_1{}^2 x}{R_1^* r_2} - \frac{k_1 \nu_2{}^2 x}{R_2^* r_2} \}$$

$$+ \frac{1}{2\pi\mu(k_1{}^2-1)(\nu_1-\nu_2)} \{ - \frac{k_1 \nu_1{}^2 \nu_2 x}{Q_1^* q_1} - \frac{k_1 \nu_1 \nu_2{}^2 x}{Q_2^* q_2} + \frac{k_1{}^2 \nu_1{}^2 \nu_2 x}{M_1^* m_1} + \frac{\nu_1 \nu_2{}^2 x}{N_2^* n_2} \}$$

$$(5.1.4)$$

$$
\begin{aligned}
\sigma_{xz} = {} & \frac{1}{4\pi(k_1-1)} \{ - \frac{\nu_1{}^2}{Q_1{}^{*} q_1} + \frac{\nu_1{}^4 x^2}{Q_1{}^{*2} q_1{}^2} + \frac{\nu_1{}^4 x^2}{Q_1{}^{*3} q_1} + \frac{k_1 \nu_2{}^2}{Q_2{}^{*2} q_2} \\
& - \frac{k_1 \nu_2{}^4 x^2}{Q_2{}^{*2} q_2{}^2} - \frac{k_1 \nu_2{}^4 x^2}{Q_2{}^{*3} q_2} + \frac{\nu_1{}^2}{R_1{}^{*} r_1} - \frac{\nu_1{}^4 x^2}{R_1{}^{*2} r_1{}^2} \\
& - \frac{\nu_1{}^4 x^2}{R_1{}^{*3} r_1} - \frac{k_1 \nu_2{}^2}{R_2{}^{*} r_2} + \frac{k_1 \;{}_2{}^4 x^2}{R_2{}^{*2} r_2{}^2} + \frac{k_1 \nu_2{}^4 x^2}{R_2{}^{*3} r_2} \} \\
& + \frac{1}{2\pi(k_1-1)(\nu_1-\nu_2)} \{ - \frac{\nu_1{}^2 \nu_2}{Q_1{}^{*} q_1} + \frac{\nu_1{}^4 \nu_2 x^2}{Q_1{}^{*2} q_1{}^2} + \frac{\nu_1{}^4 \nu_2 x^2}{Q_1{}^{*3} q_1} - \frac{k_1 \nu_1 \nu_2{}^2}{Q_2{}^{*} q_2} \\
& + \frac{k_1 \nu_1 \nu_2{}^4 x^2}{Q_2{}^{*2} q_2{}^2} + \frac{k_1 \nu_1 \nu_2{}^4 x^2}{Q_2{}^{*3} q_2} + \frac{k_1 \nu_1{}^2 \nu_2}{M_1{}^{*} m_1} - \frac{k_1 \nu_1{}^4 \nu_2 x^2}{M_1{}^{*2} m_1{}^2} \\
& - \frac{k_1 \nu_1{}^4 \nu_2 x^2}{M_1{}^{*3} m_1} + \frac{\nu_1 \nu_2{}^2}{N_2{}^{*} n_2} - \frac{\nu_1 \nu_2{}^4 x^2}{N_2{}^{*2} n_2{}^2} - \frac{\nu_1 \nu_2{}^4 x^2}{N_2{}^{*3} n_2} \} \\
& + \frac{1}{4\pi} \{ - \frac{\nu_3{}^2}{Q_3{}^{*} q_3} + \frac{\nu_3{}^4 y^2}{Q_3{}^{*2} q_3{}^2} + \frac{\nu_3{}^4 y^2}{Q_3{}^{*3} q_3} - \frac{\nu_3{}^2}{R_3{}^{*} r_3} + \frac{\nu_3{}^4 y^2}{R_3{}^{*2} r_3{}^2} \\
& + \frac{\nu_3{}^4 y^2}{R_3{}^{*3} r_3} \}
\end{aligned}
\tag{5.1.5}
$$

$$
\begin{aligned}
\sigma_{yz} = {} & \frac{1}{4\pi(k_1-1)} \{ \frac{\nu_1{}^4 xy}{Q_1{}^{*2} q_1{}^2} + \frac{\nu_1{}^4 xy}{Q_1{}^{*3} q_1} - \frac{k_1 \nu_2{}^4 xy}{Q_2{}^{*2} q_2{}^2} - \frac{k_1 \nu_2{}^4 xy}{Q_2{}^{*3} q_2} \\
& - \frac{\nu_1{}^4 xy}{R_1{}^{*2} r_1{}^2} - \frac{\nu_1{}^4 xy}{R_1{}^{*3} r_1} + \frac{k_1 \nu_2{}^4 xy}{R_2{}^{*2} r_2{}^2} + \frac{k_1 \nu_2{}^4 xy}{R_2{}^{*3} r_2} \}
\end{aligned}
$$

$$+ \frac{1}{2\,\pi(k_1-1)(\nu_1-\nu_2)}\left\{\frac{\nu_1{}^4\nu_2 xy}{Q_1{}^{*2}q_1{}^2} + \frac{\nu_1{}^4\nu_2 xy}{Q_1{}^{*2}q_1} + \frac{k_1\nu_1\nu_2{}^4xy}{Q_2{}^{*2}q_2{}^2} + \frac{k_1\nu_1\nu_2{}^4xy}{Q_2{}^{*3}q_2}\right.$$

$$\left. - \frac{k_1\nu_1{}^4\nu_2xy}{M_1{}^{*2}m_1{}^2} - \frac{k_1\nu_1{}^4\nu_2xy}{M_1{}^{*3}m_1} - \frac{\nu_1\nu_2{}^4xy}{N_2{}^{*2}n_2{}^2} - \frac{\nu_1\nu_2{}^4xy}{N_2{}^{*3}n_2}\right\}$$

$$+ \frac{1}{4\pi}\left\{- \frac{\nu_3{}^4xy}{Q_2{}^{*3}q_3{}^2} - \frac{\nu_3{}^4xy}{Q_3{}^{*3}q_3} - \frac{\nu_3{}^4xy}{R_3{}^{*2}r_3{}^2} - \frac{\nu_3{}^4xy}{R_3{}^{*3}r_3}\right\} \qquad (5.1.6)$$

$$\sigma_{zz} = \frac{1}{4\pi(k_1-1)}\left\{\frac{\nu_1{}^4x}{Q_1{}^{*3}} - \frac{k_1\nu_2{}^4x}{Q_2{}^{*3}} - \frac{\nu_1{}^4x}{R_1{}^{*3}} + \frac{k_1\nu_2{}^4x}{R_2{}^{*3}}\right\}$$

$$+ \frac{1}{2\pi(k_1-1)(\nu_1-\nu_2)}\left\{\frac{\nu_1{}^4\nu_2x}{Q_1{}^{*3}} + \frac{k_1\nu_1\nu_2{}^4x}{Q_2{}^{*3}}\right.$$

$$\left. - \frac{k_1\nu_1{}^4\nu_2x}{M_1{}^{*3}} - \frac{\nu_1\nu_2{}^4x}{N_2{}^{*3}}\right. \qquad (5.1.7)$$

$$\sigma_{xx} = \frac{1}{4\pi(k_1-1)}\left\{- \frac{\nu_1{}^2x}{Q_1{}^{*3}} + \frac{k_1\nu_2{}^2x}{Q_2{}^{*3}} + \frac{\nu_1{}^2x}{R_1{}^{*3}} - \frac{k_1\nu_2{}^2x}{R_2{}^{*3}}\right\}$$

$$+ \frac{1}{2\pi(k_1-1)(\nu_1-\nu_2)}\left\{- \frac{\nu_1{}^2\nu_2x}{Q_1{}^{*3}} - \frac{k_1\nu_1\nu_2{}^2x}{Q_2{}^{*3}} + \frac{k_1\nu_1{}^2\nu_2x}{M_1{}^{*3}} + \frac{\nu_1\nu_2{}^2x}{N_2{}^{*3}}\right\}$$

$$+ \frac{1}{2\pi\nu_3{}^2(k_1{}^2-1)}\left\{- \frac{2\nu_1{}^6xy^2}{Q_1{}^{*2}q_1{}^3} + \frac{\nu_1{}^4x}{Q_1{}^{*}q_1{}^2} - \frac{\nu_1{}^6xy^2}{Q_1{}^{*3}q_1{}^2} + \frac{2k_1{}^2\nu_2{}^6xy^2}{Q_2{}^{*2}q_2{}^3}\right.$$

$$- \frac{k_1{}^2\nu_2{}^4x}{Q_2{}^{*}\,q_2{}^2} + \frac{k_1{}^2\nu_2{}^6xy^2}{Q_2{}^{*3}q_2{}^2} + \frac{2\nu_1{}^6xy^2}{R_1{}^{*2}r_1{}^3} - \frac{\nu_1{}^4x}{R_1{}^{*}\,r_1{}^2}$$

$$+\frac{\nu_1{}^6xy^2}{R_1{}^{*3}r_1{}^2}-\frac{2k_1{}^2\nu_2{}^6xy^2}{R_2{}^{*2}r_2{}^3}+\frac{k_1{}^2\nu_2{}^4x}{R_2{}^{*}\,r_2{}^2}-\frac{k_1{}^2\nu_2{}^6xy^2}{R_2{}^{*3}r_2{}^2}\}$$

$$+\frac{1}{\pi\nu_3{}^2(k_1{}^2-1)(\nu_1-\nu_2)}\{-\frac{2\nu_1{}^6\nu_2xy^2}{Q_1{}^{*2}q_1{}^3}+\frac{\nu_1{}^4\nu_2x}{Q_1{}^{*}\,q_1{}^2}-\frac{\nu_1{}^6\nu_2xy^2}{Q_1{}^{*3}q_1{}^2}-\frac{2k_1{}^2\nu_1\nu_2{}^6xy^2}{Q_2{}^{*2}q_2{}^3}$$

$$+\frac{k_1{}^2\nu_1\nu_2{}^4x}{Q_2{}^{*}\,q_2{}^2}-\frac{k_1{}^2\nu_1\nu_2{}^6xy^2}{Q_2{}^{*3}q_2{}^2}+\frac{2k_1\nu_1{}^6\nu_2xy^2}{M_1{}^{*2}m_1{}^3}-\frac{k_1\nu_1{}^4\nu_2x}{M_1{}^{*}\,m_1{}^2}$$

$$+\frac{k_1\nu_1{}^6\nu_2xy^2}{M_1{}^{*3}m_1{}^2}+\frac{2k_1\nu_1\nu_2{}^6xy^2}{N_2{}^{*2}n_2{}^3}-\frac{k_1\nu_1\nu_2{}^4x}{N_2{}^{*}n_2{}^2}+\frac{k_1\nu_1\nu_2{}^6xy^2}{N_2{}^{*3}n_2{}^2}\}$$

$$+\frac{1}{2\pi}\{\frac{2\nu_3{}^4xy^2}{Q_3{}^{*2}q_3{}^3}-\frac{\nu_3{}^2\,x}{Q_3{}^{*}\,q_3{}^2}+\frac{\nu_3{}^4xy^2}{Q_3{}^{*3}q_3{}^2}$$

$$+\frac{2\nu_3{}^4xy^2}{R_3{}^{*2}r_3{}^3}-\frac{\nu_3{}^2x}{R_3{}^{*}r_3{}^2}+\frac{\nu_3{}^4xy^2}{R_3{}^{*3}r_3{}^2}\} \qquad (5.1.8)$$

$$\sigma_{yy}=\frac{1}{4\pi(k_1-1)}\{-\frac{\nu_1{}^2x}{Q_1{}^{*3}}+\frac{k_1\nu_2{}^2x}{Q_2{}^{*3}}+\frac{\nu_1{}^2x}{R_1{}^{*3}}-\frac{k_1\nu_2{}^2x}{R_2{}^{*3}}\}$$

$$+\frac{1}{2\pi(k_1-1)(\nu_1-\nu_2)}\{-\frac{\nu_1{}^2\nu_2x}{Q_1{}^{*3}}-\frac{k_1\nu_1\nu_2{}^2x}{Q_2{}^{*3}}+\frac{k_1\nu_1{}^2\nu_2x}{M_1{}^{*3}}+\frac{\nu_1\nu_2{}^2x}{N_2{}^{*3}}\}$$

$$+\frac{1}{2\pi\nu_3{}^2(k_1{}^2-1)}\{-\frac{2\nu_1{}^6x^3}{Q_1{}^{*2}q_1{}^3}+\frac{3\nu_1{}^4x}{Q_1{}^{*}\,q_1{}^2}-\frac{\nu_1{}^6x^3}{Q_1{}^{*3}q_1{}^2}+\frac{2k_1{}^2\nu_2{}^6x^3}{Q_2{}^{*2}q_2{}^3}$$

$$-\frac{3k_1{}^2\nu_2{}^4x}{Q_2{}^{*}q_2{}^2}+\frac{k_1{}^2\nu_2{}^6x^3}{Q_2{}^{*3}q_2{}^2}+\frac{2\nu_1{}^6x^3}{R_1{}^{*2}r_1{}^3}-\frac{3\nu_1{}^4x}{R_1{}^{*}r_1{}^2}$$

$$+\frac{\nu_1{}^6x^3}{R_1{}^{*3}r_1{}^2}-\frac{2k_1{}^2\nu_2{}^6x^3}{R_2{}^{*2}r_2{}^3}+\frac{3k_1{}^2\nu_2{}^4x}{R_2{}^{*}\,r_2{}^2}-\frac{k_1{}^2\nu_2{}^6x^3}{R_2{}^{*3}r_2{}^2}\}$$

$$+\frac{1}{\pi\nu_3{}^2(k_1{}^2-1)(\nu_1-\nu_2)}\{-\frac{2\nu_1{}^6\nu_2x^3}{Q_1{}^{*2}q_1{}^3}+\frac{3\nu_1{}^4\nu_2x}{Q_1{}^{*}\,q_1{}^2}-\frac{\nu_1{}^6\nu_2x^3}{Q_1{}^{*3}q_1{}^2}-\frac{2k_1{}^2\nu_1\nu_2{}^6x^3}{Q_2{}^{*2}q_2{}^3}$$

$$+\frac{3k_1{}^2\nu_1\nu_2{}^4x}{Q_2{}^{*}\,q_2{}^2}-\frac{k_1{}^2\nu_1\nu_2{}^6x^3}{Q_2{}^{*3}q_2{}^2}+\frac{2k_1\nu_1{}^6\nu_2x^3}{M_1{}^{*2}m_1{}^3}-\frac{3k_1\nu_1{}^4\nu_2x}{M_1{}^{*}\,m_1{}^2}$$

$$+\frac{k_1\nu_1{}^6\nu_2x^3}{M_1{}^{*3}m_1{}^2}+\frac{2k_1\nu_1\nu_2{}^6x^3}{N_2{}^{*2}n_2{}^3}-\frac{3k_1\nu_1\nu_2{}^4x}{N_2{}^{*}n_2{}^2}+\frac{k_1\nu_1\nu_2{}^6x^3}{N_2{}^{*3}n_2{}^2}\}$$

$$+\frac{1}{2\pi}\{-\frac{2\nu_3{}^4xy^2}{Q_3{}^{*2}q_3{}^3}+\frac{\nu_3{}^2x}{Q_3{}^{*}q_3{}^2}-\frac{\nu_3{}^4xy^2}{Q_3{}^{*3}q_3{}^2}$$

$$-\frac{2\nu_3{}^4xy^2}{R_3{}^{*2}r_3{}^3}+\frac{\nu_3{}^2x}{R_3{}^{*}r_3{}^2}-\frac{\nu_3{}^4xy^2}{R_3{}^{*3}r_3{}^2}\} \qquad (5.1.9)$$

$$\sigma_{xy}=\frac{1}{2\pi\nu_3{}^2(k_1{}^2-1)}\{\frac{2\nu_1{}^6x^2y}{Q_1{}^{*2}q_1{}^3}-\frac{\nu_1{}^4y}{Q_1{}^{*}q_1{}^2}+\frac{\nu_1{}^6x^2y}{Q_1{}^{*3}q_1{}^2}-\frac{2k_1{}^2\nu_2{}^6x^2y}{Q_2{}^{*2}q_2{}^3}$$

$$+\frac{k_1{}^2\nu_2{}^4y}{Q_2{}^{*}q_2{}^2}-\frac{k_1{}^2\nu_2{}^6x^2y}{Q_2{}^{*3}q_2{}^2}-\frac{2\nu_1{}^6x^2y}{R_1{}^{*2}r_1{}^3}+\frac{\nu_1{}^4y}{R_1{}^{*}r_1{}^2}$$

$$-\frac{\nu_1{}^6x^2y}{R_1{}^{*3}r_1{}^2}+\frac{2k_1{}^2\nu_2{}^6x^2y}{R_2{}^{*2}r_2{}^3}-\frac{k_1{}^2\nu_2{}^4y}{R_2{}^{*}r_2{}^2}+\frac{k_1{}^2\nu_2{}^6x^2y}{R_2{}^{*3}r_2{}^2}\}$$

$$+\frac{1}{\pi\nu_3{}^2(k_1{}^2-1)(\nu_1-\nu_2)}\{\frac{2\nu_1{}^6\nu_2x^2y}{Q_1{}^{*2}q_1{}^3}-\frac{\nu_1{}^4\nu_2y}{Q_1{}^{*}q_1{}^2}+\frac{\nu_1{}^6\nu_2x^2y}{Q_1{}^{*3}q_1{}^2}+\frac{2k_1{}^2\nu_1\nu_2{}^6x^2y}{Q_2{}^{*2}q_2{}^3}$$

$$- \frac{k_1{}^2\nu_1\nu_2{}^4y}{Q_2{}^*q_2{}^2} + \frac{k_1{}^2\nu_1\nu_2{}^6\nu x^2y}{Q_2{}^{*3}q_2{}^2} - \frac{2k_1\nu_1{}^6\nu_2\ x^2y}{M_1{}^{*2}m_1{}^3} + \frac{k_1\ {}_1{}^4\nu_2y}{M_1{}^*m_1{}^2}$$

$$- \frac{k_1\nu_1{}^6\nu_2x^2y}{M_1{}^{*3}m_1{}^2} - \frac{2k_1\nu_1\nu_2{}^6x^2y}{N_2{}^{*2}n_2{}^3} + \frac{k_1\nu_1\nu_2{}^4y}{N_2{}^*n_2{}^2} - \frac{k_1\nu_1\nu_2{}^6x^2y}{N_2{}^{*3}n_2{}^2} \}$$

$$+ \frac{1}{4\pi\nu_3{}^2} \{- \frac{2\nu_3{}^6y(x^2-y^2)}{Q_3{}^{*2}q_3{}^3} - \frac{2\nu_3{}^4y}{Q_3{}^*q_3{}^2} - \frac{\nu_3{}^6y(x^2-y^2)}{Q_3{}^{*3}q_3{}^2}$$

$$- \frac{2\nu_3{}^6y(x^2-y^2)}{R_3{}^{*2}r_3{}^3} - \frac{2\nu_3{}^4y}{R_3{}^*r_3{}^2} - \frac{\nu_3{}^6y(x^2-y^2)}{R_3{}^{*3}r_3{}^2} \} \qquad (5.1.10)$$

5.2 Discussion and Plots

The behavior on the surface of a transversely isotropic half-space with a horizontal force acting a point below the surface is of particular interest to the observer.

It must be noted that the approach to the surface must be effected uniformly through the medium, that is, one first allows x and y to approach their value and then, lets z approach zero. In so doing one eliminates the possibility of taking readings in the upper half-space which contains a number of singularities and will yield erroneous results.

In order to more clearly see the effect of transverse isotropy, the normal displacement U_3 at the surface $z = 0$ were first plotted for an isotropic medium with a Poisson ratio, ν of .25. This was done by allowing ν_1 to approach ν_2 and $k_1=k_2$ to approach 1 and by realizing that functions of the elastic constants for transverse isotropy may approach functions of isotropic elastic constants. For example, the factor $\frac{k_1-1}{\nu_1-\nu_2}$ approaches $2(1-\nu)$ as a limit when $k_1=k_2$ approaches unity. The resulting vertical displacement is plotted in Figure 5.2.

In order to compare the isotropic results with those for meaningful transversely isotropic media, the three elements mentioned in Chapter 1 were scrutinized. The values for the constants required are tabulated in Table 5.1 below.

Table 1.5 CONSTANT PARAMETERS FOR SELECTED MATERIALS

	MAGNESIUM	ZINC	CADMIUM
k_1	2.7305	.1236 + i.9923	.8236 + .5672
k_2	.3662	.1236 - i.9923	.8236 - i.5672
ν_1	1.4088	.6781 + i.4006	.7856 + i.1878
ν_2	.7238	.6781 - i.4006	.7856 - i.1878
ν_3	1.0042	.7437	.7149
$\nu_1{}^2$	1.9848	.2993 + i.5433	.5819 + i.2950
$\nu_2{}^2$	.5238	.2993 - i.5433	.5819 - i.2950
$\nu_3{}^2$	1.0084	.5530	.5111

Interestingly all the values for magnesium are real while certain values for both zinc and cadmium contain both a real and an imaginary component. This fact is not disturbing when it is realized that magnesium is the most nearly isotropic of the three materials.

When comparing the vertical displacements experienced by the three materials with the isotropic case, not much behavioral difference (see Figures 5.3, 5.4, and 5.5 is noted. Therefore, in an attempt to get a better physical grasp and to cast some light on the behavior of such transversely isotropic materials, it was decided to examine the stress σ_{yy} ("hoop stress") at the surface. Again, the isotropic case (ν=.25) was examined first (see Figure 5.6) and compared to the other three materials (see Figures 5.7, 17 and 18). As expected the nearly-isotropic magnesium did not display radically different behavior as compared to the isotropic case. However both zinc and cadmium behaved differently for the lower $\frac{r}{s}\cos\theta$ ratios. This again may be due to the behavior noted for these materials in Chapter 1.

As the ratio $\frac{r}{s}\cos\theta$ got significantly larger, the "hoop stress" approached the asymtotic values listed below:

ISOTROPY (ν=.25)	$-.1592\ r^2$
MAGNESIUM	$-.3170\ r^2$
ZINC	$-.4779\ r^2$
CADMIUM	$-.4452\ r^2$

A much more extensive numerical study of this problem solution, as well as those of Chapter 4, is justified. Such in-depth analysis was felt to be beyond the scope of the present investigation.

6. SUMMARY AND RECOMMENDATIONS FOR FURTHER STUDY

6.1 Summary

In this study the method of singularities has been extended to solve the problem of a concentrated tangential unit force applied beneath the surface of a transversely isotropic medium. Additionally, the previously-solved transversely isotropic analogs of the Boussinesq, Cerruti and Mindlin problems were solved by extending the same method of singularities.

The method of singularities has proven to be a fairly direct and useful approach to the solution of the basic problems in a transversely isotropic medium. The relative simplicity of generating a series of singularities and their superposition to solve substantially more intricate problems makes computations a simple matter. Moreover, some physical meaning can be attached to each stage of the solution process.

6.2 Recommendations for Further Study

Several possible extensions of the present work immediately present themselves.

1. The literature contains virtually no specific numerical results for the previously published singular solutions which are reviewed in Chapter 4. Indeed, Chapter 4 contains the only available complete presentation of the displacements and stresses for these solutions. Clearly, an extensive numerical study of the solutions of Chapter 4, together with the new solution presented here, can be of value to engineers who need a more

complete understanding of ways in which the behavior of anisotropic materials can differ from that of isotropic materials.

2. Existing singular solutions for a transversely isotropic half-space orient the surface to be parallel to the "isotropic planes" (x-y planes) in the medium. The presented approach of generating a solution by superposition of physically interpretable singularities should permit the direct generation of half-space solutions for an arbitrary orientation of the surface. Although such solutions will certainly be more complex than those presented here, they will apparently be in closed form.

3. It has been shown by Eubank and Sternberg (1954) that the fundamental singularity for a concentrated force on a curved surface of an isotropic medium differs from that of the Boussinesq problem. Singularities of lower order are essential for the satisfaction of the boundary condition "in the small". This study can be duplicated for transverse isotropy; the explicit way in which the material behavior affects this phenomenon should be of more than passing interest.

4. Several general problem solutions in isotropic elasticity theory, such as "the problem of the spherical cavity" can be interpreted as a superposition of singular solutions of the equations of elasticity theory. The analogs of these problems for a transversely isotropic medium should yield to extensions of the procedures which are used in this report.

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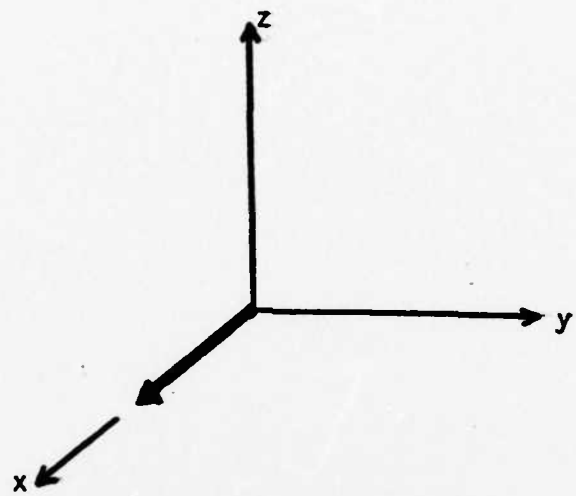


FIG. 3.1 HORIZONTAL UNIT FORCE-INFINITE MEDIUM

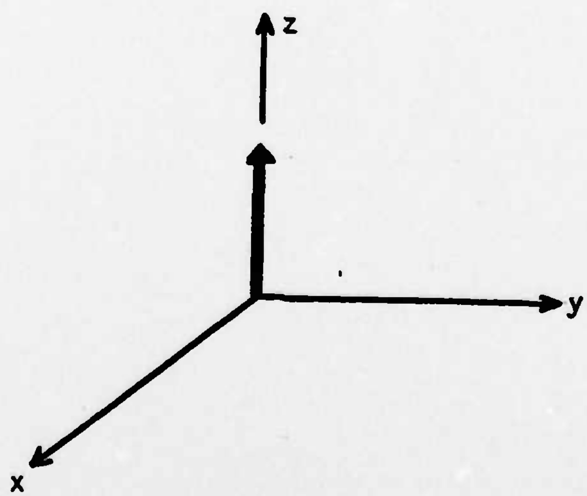
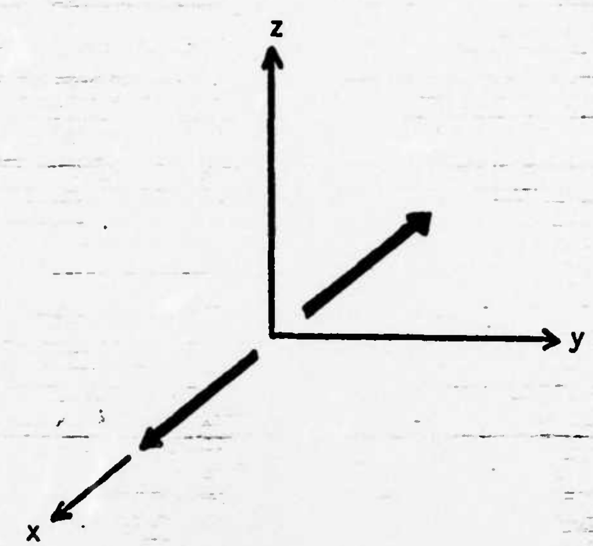
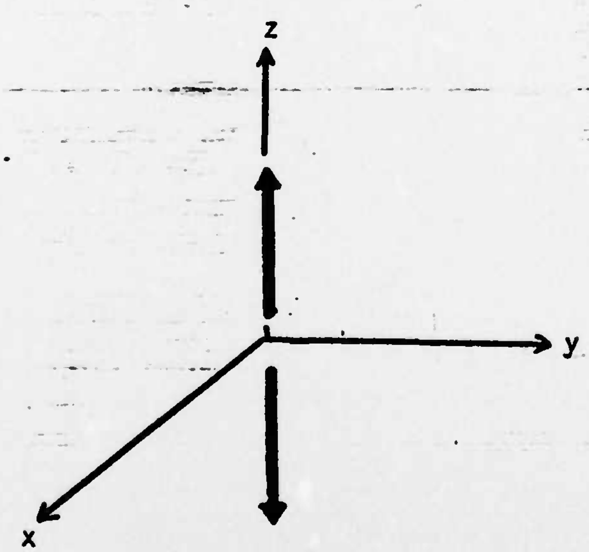


FIG. 3.2 VERTICAL UNIT FORCE-INFINITE MEDIUM



(a) x-Direction



(b) z-Direction

FIG. 3.3 DOUBLE FORCES WITHOUT MOMENT

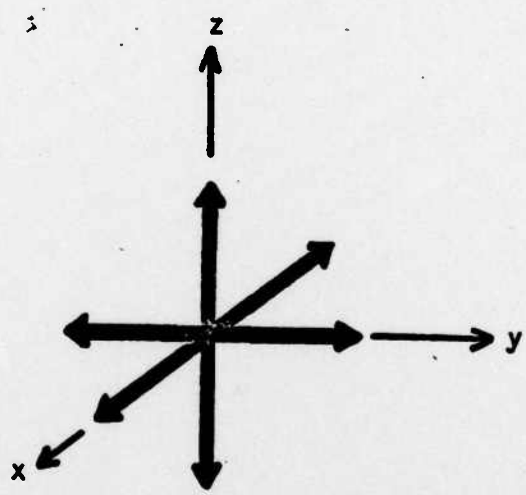
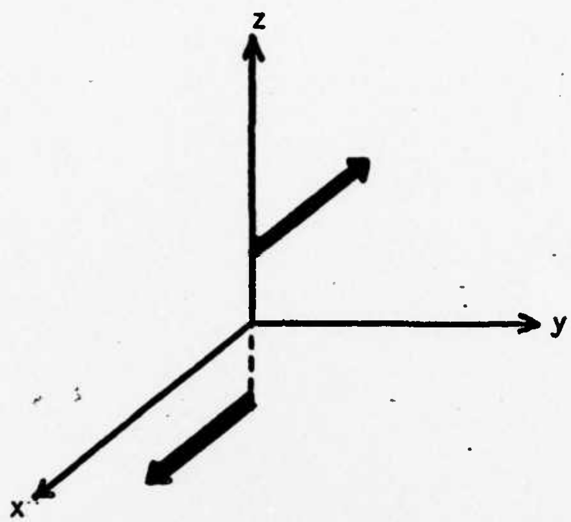
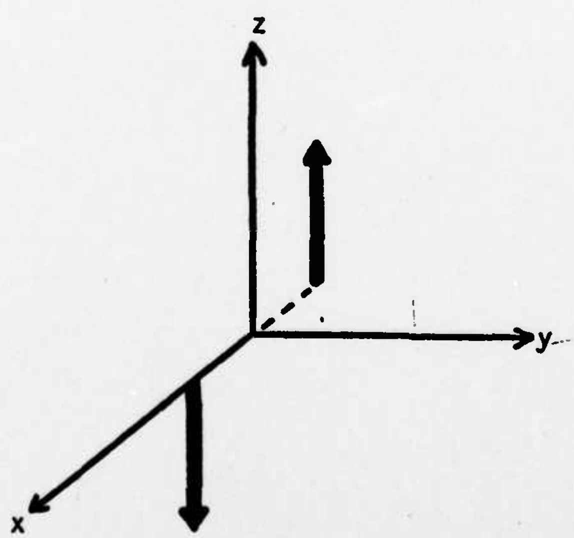


FIG. 3.4 CENTER OF DILATATION



(a) x-Direction



(b) z-Direction

FIG. 3.5 DOUBLE FORCES WITH MOMENTS ABOUT y-AXIS

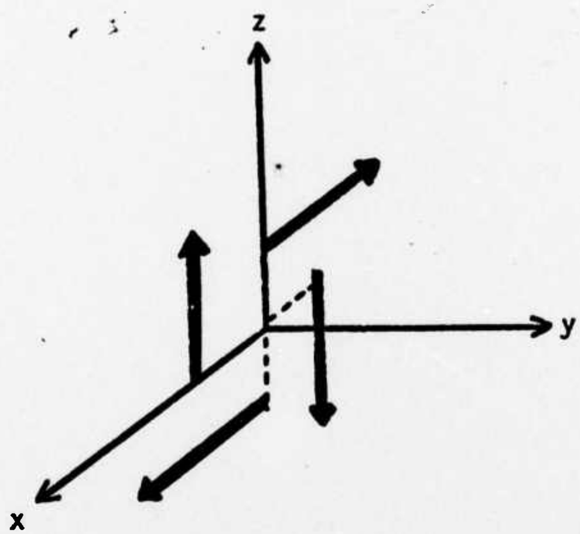


FIG 3.6 CENTER OF ROTATION ABOUT y-AXIS

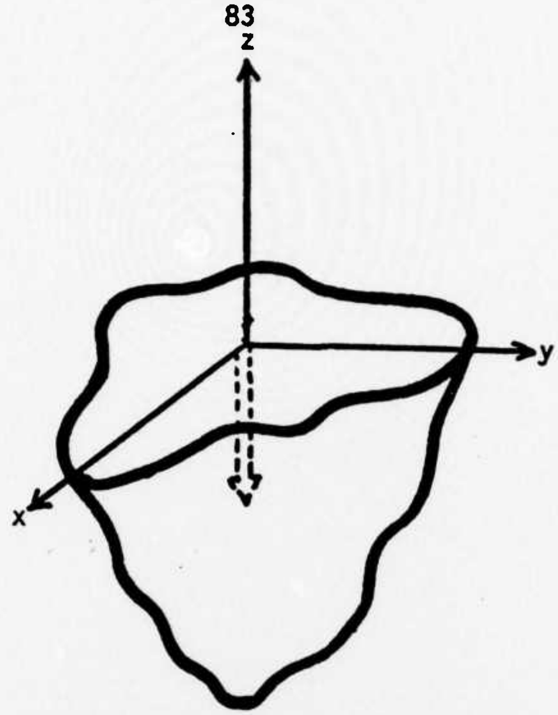


FIG. 4.1 BOUSSINESQ PROBLEM-TRANSVERSE ISOTROPY

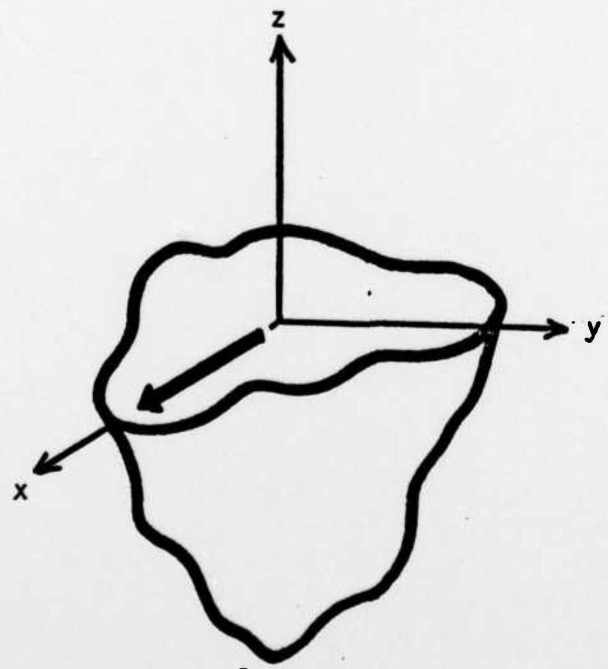


FIG. 4.2 CERRUTI PROBLEM-TRANSVERSE ISOTROPY

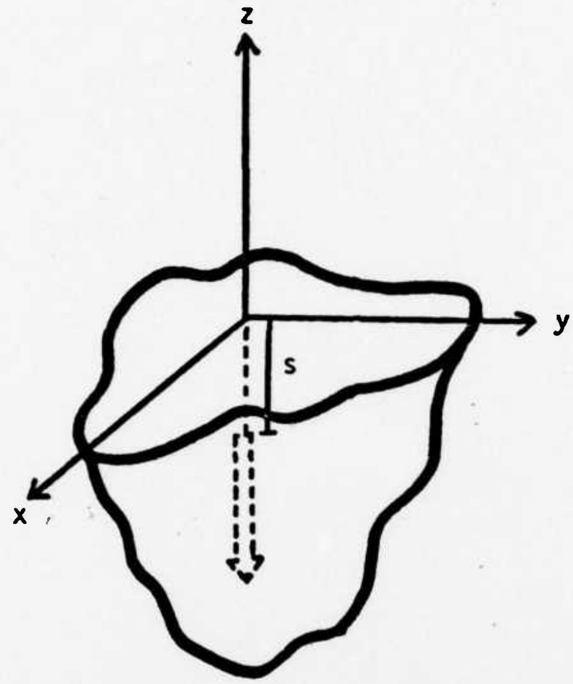


FIG 4.3 MINDLIN PROBLEM-TRANSVERSE ISOTROPY

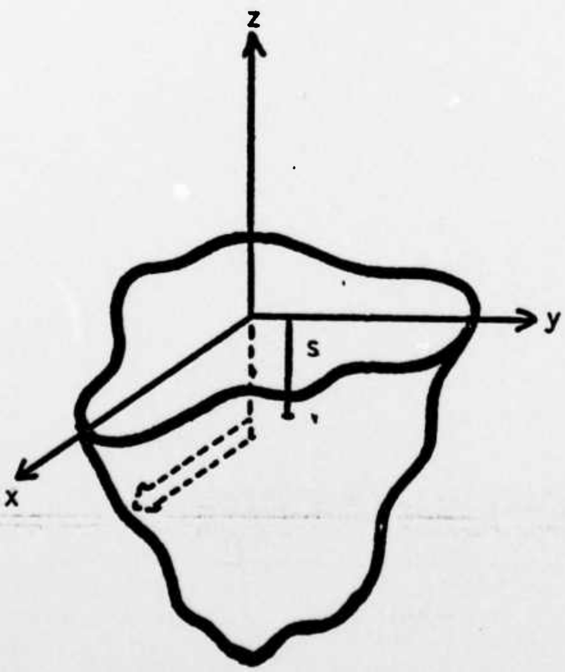


FIG. 5.1 TANGENTIAL UNIT FORCE BENEATH THE SURFACE OF A TRANSVERSELY ISOTROPIC HALF-SPACE

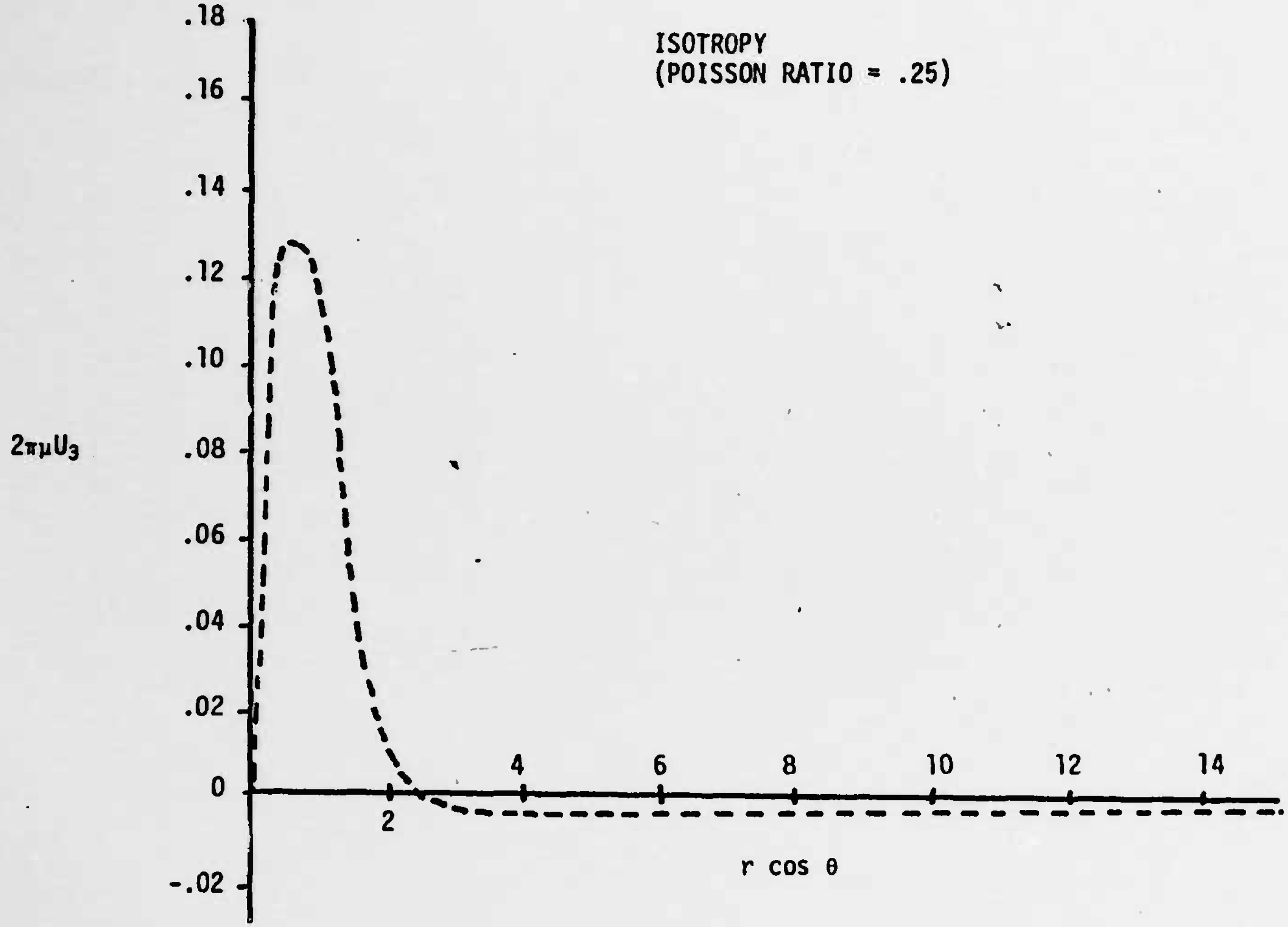


FIG. 5.2 DISPLACEMENTS-z-DIRECTION-ISOTROPY

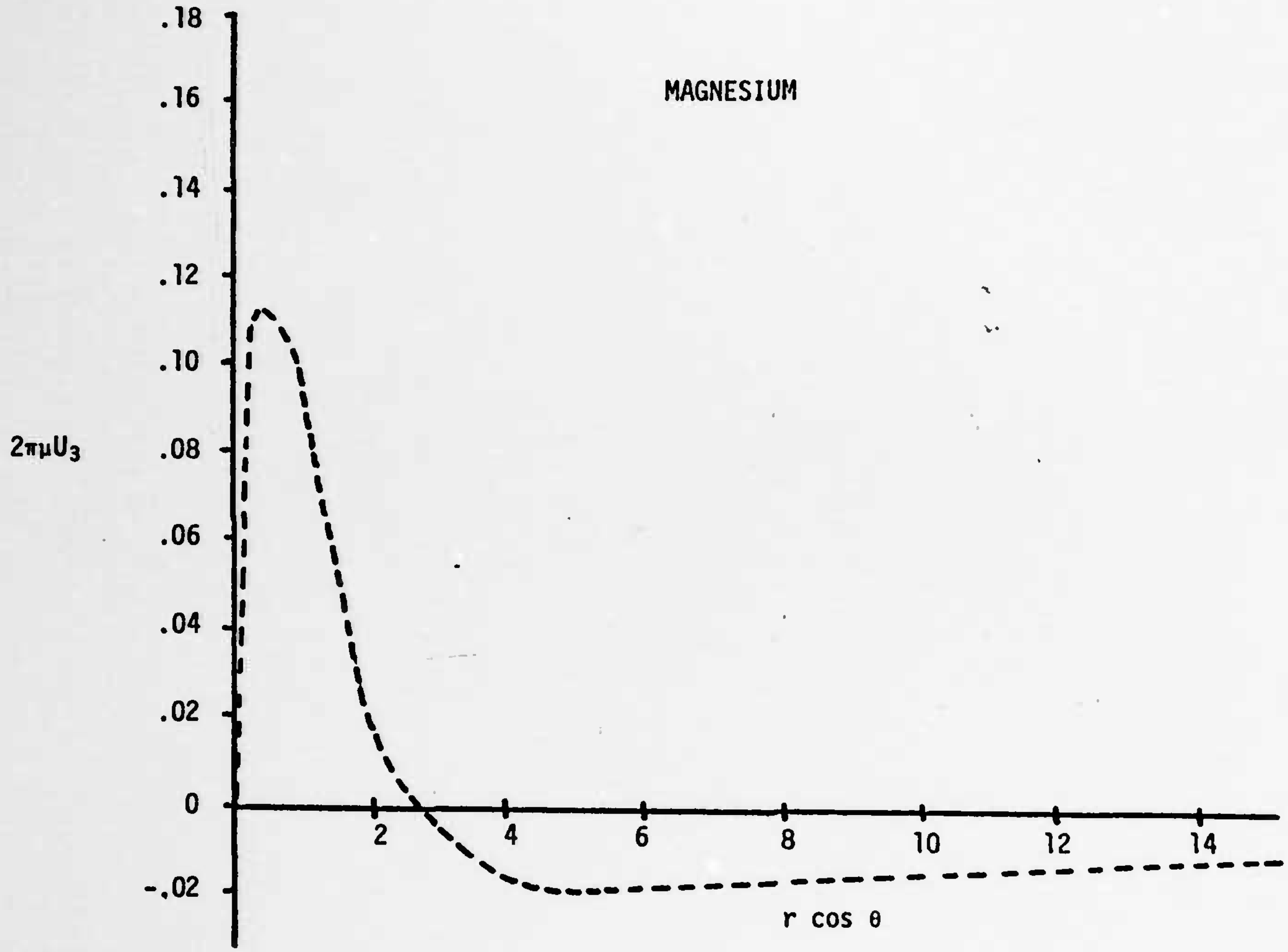


FIG. 5.3 DISPLACEMENTS-z-DIRECTION-MAGNESIUM

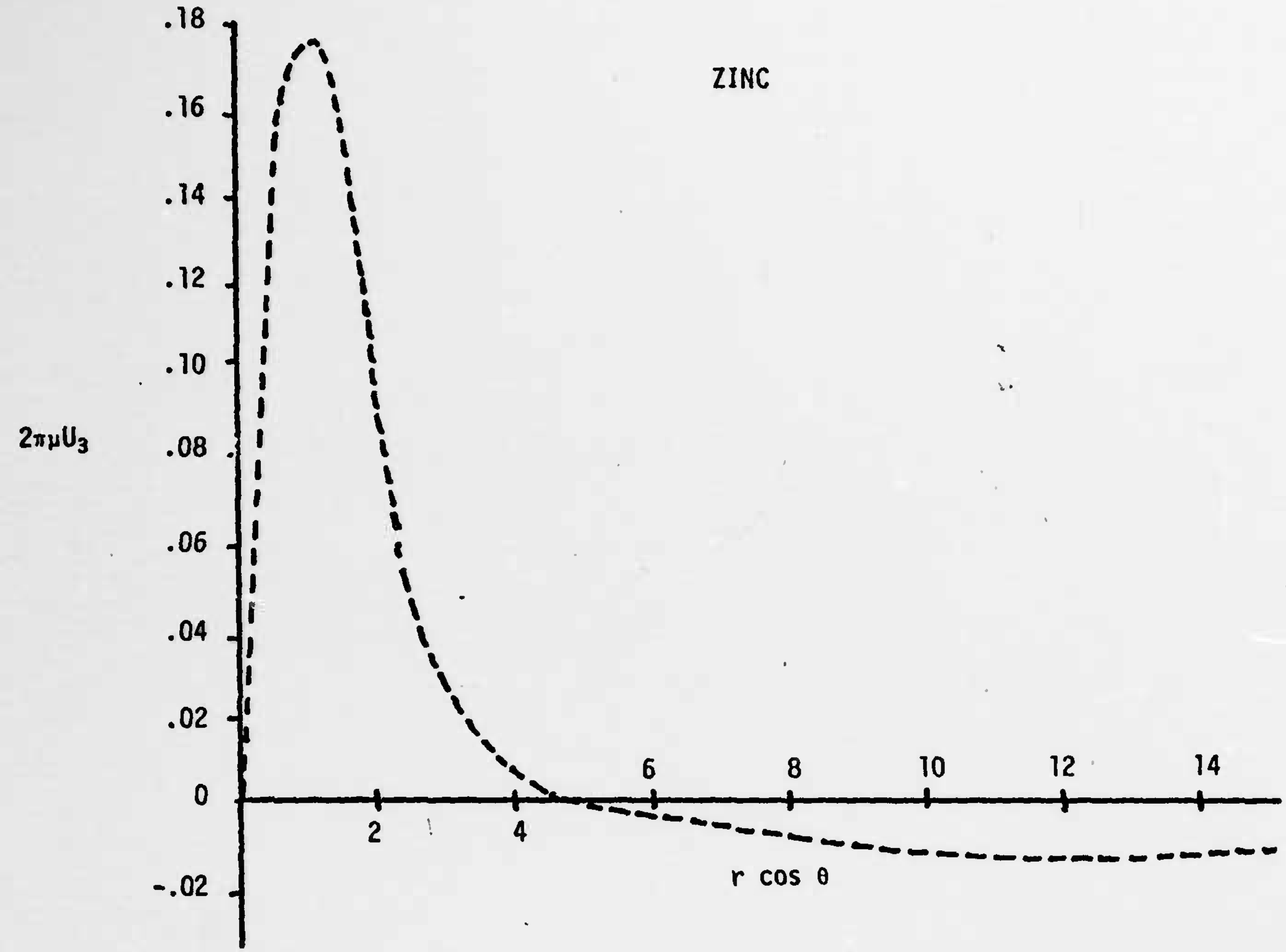


FIG. 5.4 DISPLACEMENTS-z-DIRECTION-ZINC

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CONCENTRATED FORCE PROBLEMS IN TRANSVERSE ISOTROPY.(U)
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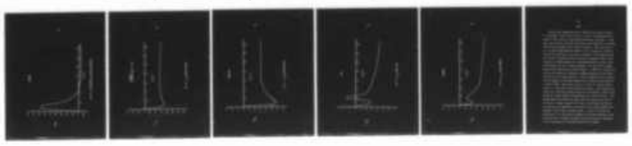

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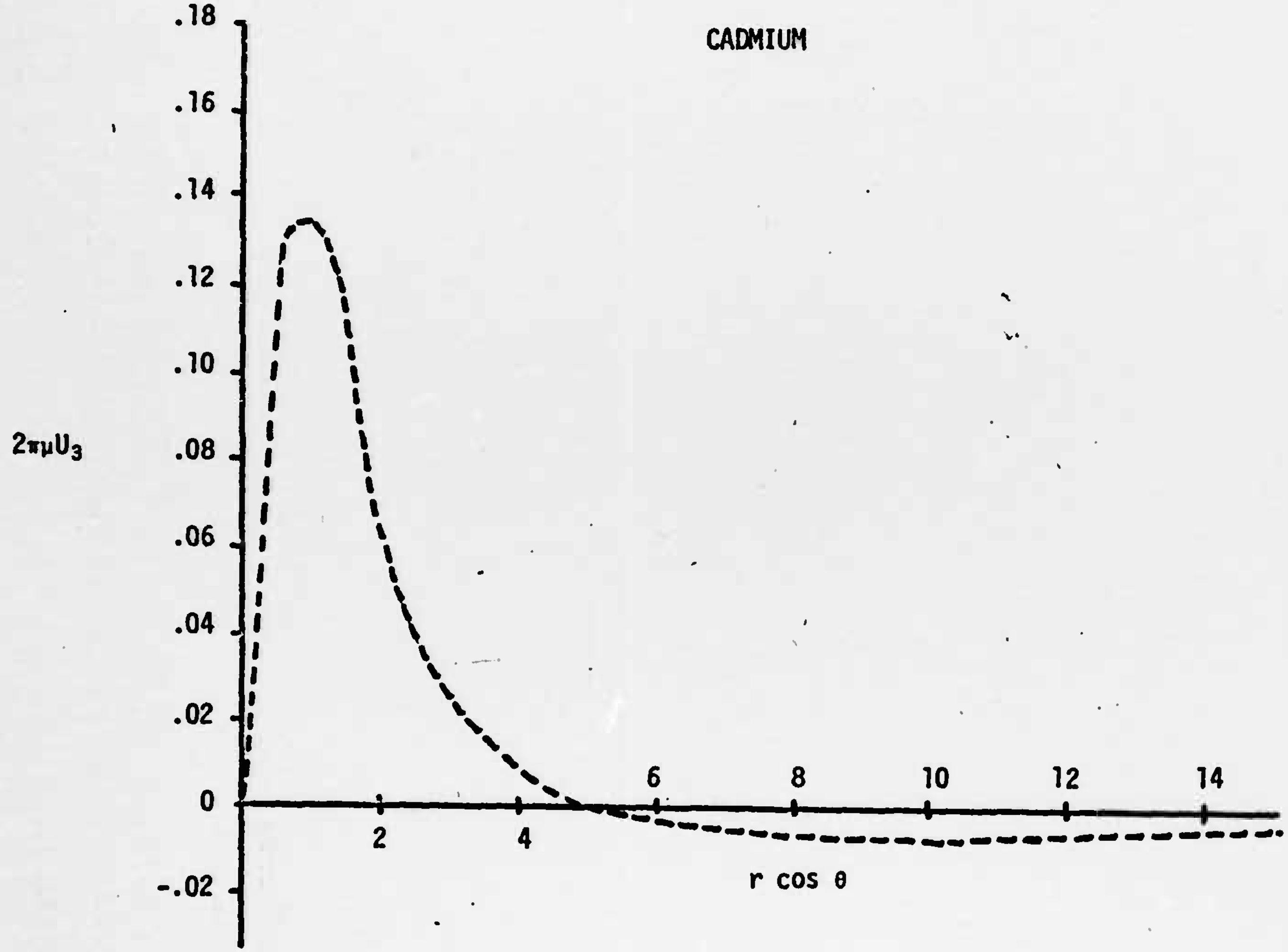

FIG. 5.5 DISPLACEMENTS-z-DIRECTION-CADMIUM

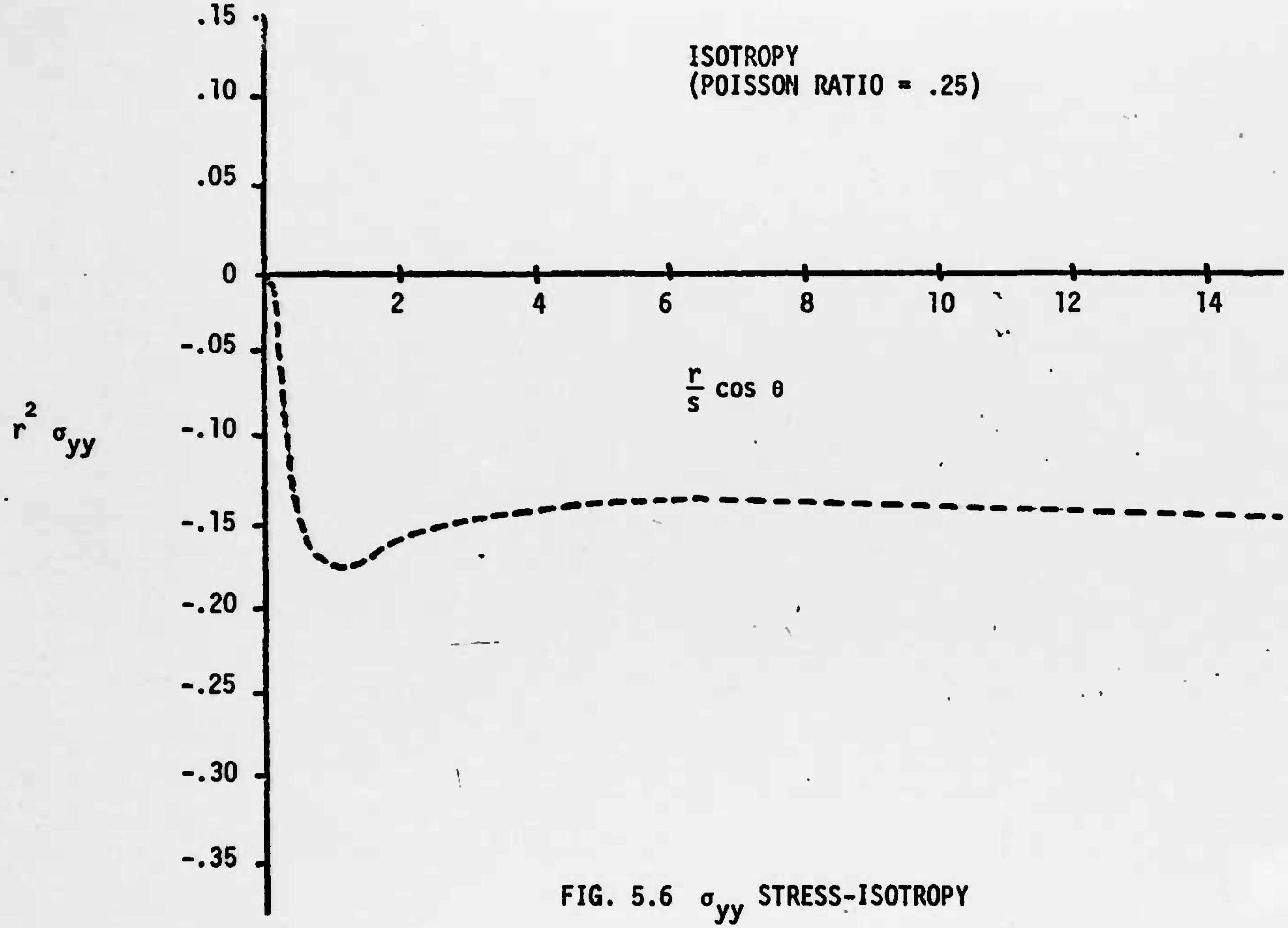

FIG. 5.6 σ_{yy} STRESS-ISOTROPY

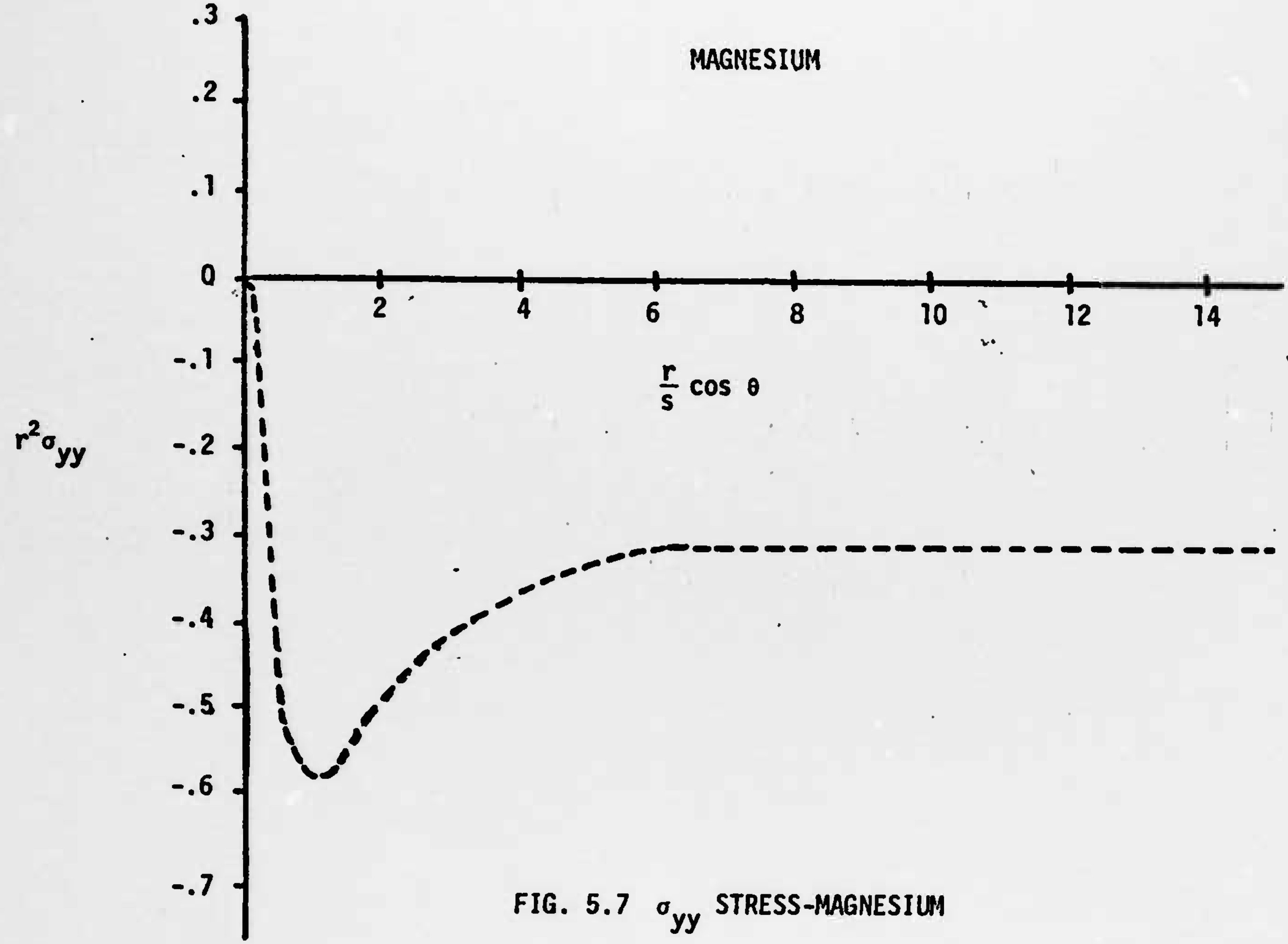

FIG. 5.7 σ_{yy} STRESS-MAGNESIUM

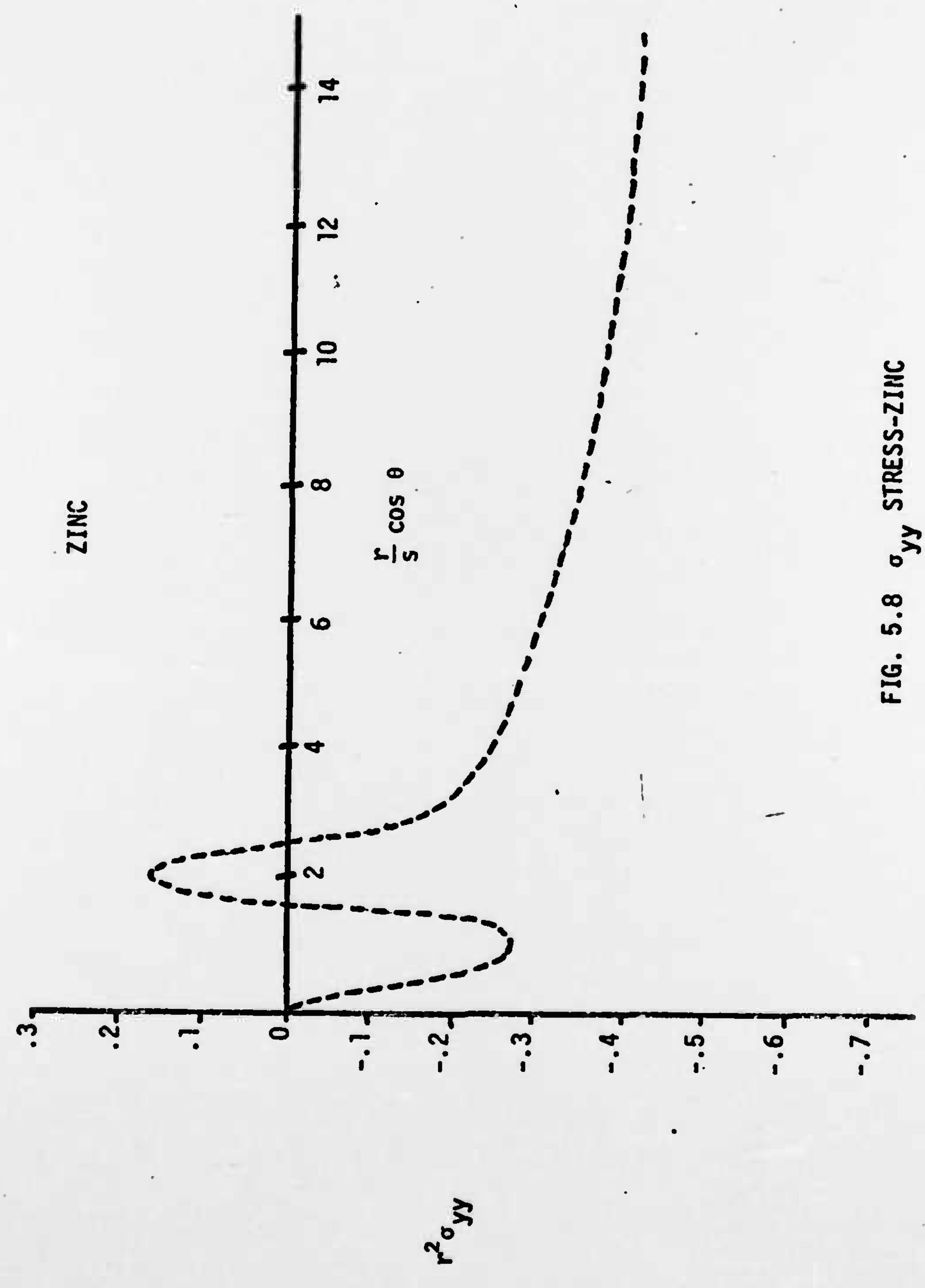

FIG. 5.8 σ_{yy} STRESS-ZINC

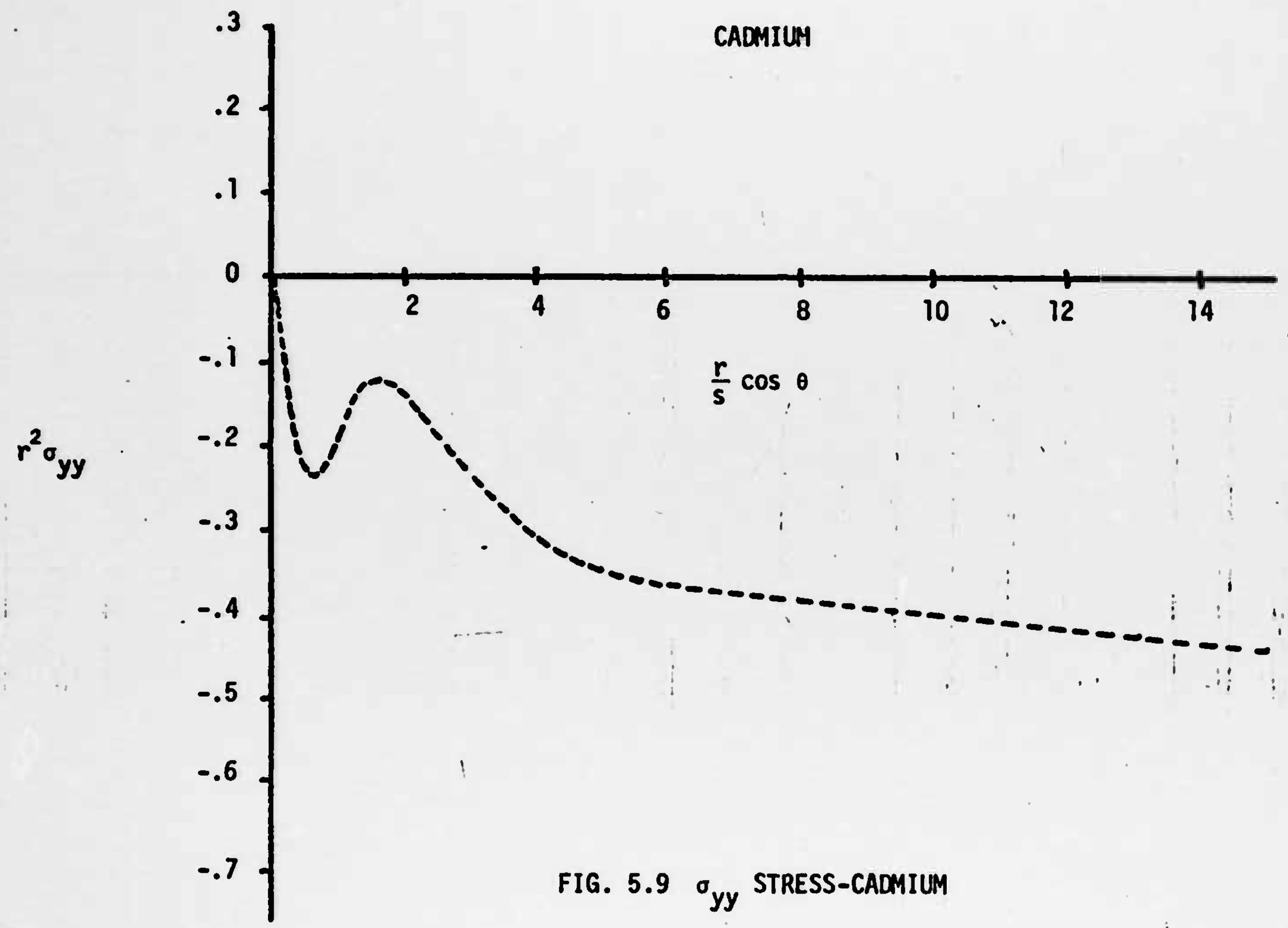

FIG. 5.9 σ_{yy} STRESS-CADMIUM

VITA

Giuliano Maria Toneatto was born in Flambro, Italy on May 3, 1946 of U.S. parentage. After arrival in the U.S. in 1947, he attended parochial schools in New York and in 1963 graduated from Xavier High School. In that year, after passing a State-wide examination, he was appointed to the United States Military Academy by Rep. Scheuer of New York. After receiving his B.S. in General Engineering with emphasis in Civil Engineering in 1967, Mr. Toneatto was commissioned in the Corps of Engineers. After completing the Basic Course for Engineer Officers, Parachute, and Ranger Training, he was assigned to a Combat Engineer unit in West Germany where he served as both Platoon Leader and later as Company Commander. His activities in Germany centered about demolitions and nuclear weapons. In 1969-70 Mr. Toneatto saw service in the Republic of Vietman as a Company Commander and later Operations Officer of a Construction unit. Under his command, the unit constructed 62 kilometers of high-speed highways which included the construction of six major bridges. Upon return to the U.S. in 1971, Mr. Toneatto studied under a U.S. Army Grant at the University of Illinois. After receiving his license as a Professional Engineer and his M.S. (Structures) in 1972, he was assigned as Project Engineer on the Washington D.C. Bulk Mail Center. This project became the prototype for twelve other such centers across the U.S. In 1974, Mr. Toneatto, because of his fluency in both German and Italian, was selected by the Department of the Army to serve as Aide-de-Camp to the Commander of Land Forces in Southern Europe. In 1977, Mr. Toneatto was selected for a U.S. Army Grant to complete his doctoral dissertation at the University with a subsequent assignment as Assistant-Professor, Department of Civil Engineering, at the United States Military Academy.

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